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MATHEMATICS

On the Moore Triodic Theorem

by

A. LELEK

Presented by K. KURATOWSKI on March 4, 1960

1. Preliminaries. A continuum is said to be *triodic* (see [2], p. 262) if it contains three subcontinua A, B and C, each of which does not contain any of others and $A \cap B = A \cap C = B \cap C = A \cap B \cap C$ is a continuum. By virtue of the well known triodic theorem of Moore [2] we have

(M) *Each uncountable collection of triodic continua lying in the plane contains at least two elements having a point in common* *).

This theorem will be generalized in the present paper: we shall find a theorem (M*) corresponding to (M) and concerning arbitrary locally connected and totally bounded metric spaces (see par. 5).

Some applications to Euclidean spaces E^n will be indicated in par. 6.

If X is a metric space with the metric ϱ, the family 2^X of non-empty closed bounded subsets of X is considered as a metric space with Hausdorff distance $d(A, B)$ for $A, B \epsilon 2^X$.

We say that the set A *separates* the set B if $B - A$ is not connected, i.e. $B - A = M \cup N$, where $M \neq 0 \neq N$ and $(\overline{M} \cap N) \cup M \cap \overline{N}) = 0$. Then we say also that A *separates* B *into the sets* M *and* N.

$C_A(B)$ denotes the component of B containing A (if such exists). The set $\mathrm{Fr}_A(B) = \overline{B} \cap \overline{A - A} \cap A$ is called a *relative boundary* of B in A.

2. A generalization of the notion of triod. We say shortly that the subset T of the space X is a τ-*set in* X if there exist a number $\varepsilon > 0$ and sets $P, Q \subset T$ such that $P \cap Q$ is a non-empty connected set and for every region (i. e. open connected set) $R \subset X$ such that $\overline{P \cap Q} \subset R$ and $d(\overline{P \cap Q}, \overline{R}) < \varepsilon$, the following conditions are satisfied:

(i) $$C_{P \cap Q}(P \cap R) \text{ separates } R,$$

(ii) $$C_{P \cap Q}(Q \cap R) \neq P \cap Q.$$

*) The theorem proved in [2] is stronger. Namely, under the same hypotheses it says that there exists an uncountable subcollection every two elements of which have a point in common. But just the version (M) has a lot of applications in the plane topology.

The set P will be called a *pith* of T.

Each triodic continuum $T \subset E^2$ is a τ-set in E^2.

Indeed, let $A, B, C \subset T$ be subcontinua given by the definition of triodic continuum (see par. 1). Then, putting $P = A \cap B$, $Q = C$ and

$$\varepsilon = \min [d(A, A \cap B), d(B, A \cap B)],$$

we include that for every region $R \subset E^2$ such that $\overline{P \cap Q} \cap R$ and $d(\overline{P \cap Q}, \overline{R}) < \varepsilon$, the conditions (i) and (ii) are satisfied, since $\overline{P \cap Q} = P \cap Q = A \cap B$.

3. Transpiercing sets. Let A and B be subsets of the space X. We say that A *transpierces* B if there exists a region $R \subset X$ such that B separates R and at least one of components of $A \cap R$ intersects two different components of $R - B$.

Of course, if A transpierces B, then $A \cap B \neq 0$, but not inversely.

The transpiercing is generally an asymmetric relation, but in particular it is symmetric for the arcs lying in the plane. Indeed, we have the following obvious property of arcs:

Let A_1 and A_2 be arcs lying in the plane. Then A_1 transpierces A_2 if and only if for every two simple closed curves C_1 and C_2 lying in the plane and such that $A_1 \subset C_1$ and $A_2 \subset C_2$, the interior of C_1 intersects the interior of C_2.

Then we say also that the arcs A_1 and A_2 are *crossed*.

4. A lemma on ordered metric spaces. The order $\prec$ of a metric space Z is called (see [3], p. 41) a *natural order* if each point $z \in Z$ separates Z into the sets $\{\zeta \in Z: \zeta \prec z\}$ and $\{\zeta \in Z: z \prec \zeta\}$.

LEMMA. *If an uncountable separable metric space Z with the metric ϱ is ordered by natural order $\prec$ and a function $f: Z \to Z$ (not necessarily continuous) is such that*

$$z \prec f(z) \quad \text{for} \quad z \in Z, \tag{1}$$

then there exists an uncountable subset $Y \subset Z$ such that

$$y' \prec f(y) \quad \text{for} \quad y, y' \in Y.$$

Proof. Since $\prec$ is a natural order, (1) implies

$$0 < \varrho(z, \{\zeta \in Z: f(z) \prec \zeta\}).$$

Then for every $z \in Z$ there exists a number $\varepsilon_z > 0$ such that

$$\text{if } \varrho(z, z') < \varepsilon_z, \quad \text{then} \quad z' \prec f(z). \tag{2}$$

But Z is uncountable. Therefore, there exist a number $\varepsilon > 0$ and an uncountable subset $Z' \subset Z$ such that $\varepsilon < \varepsilon_z$ for every $z \in Z'$. Now, Z being a separable metric space, Z' must have a condensation point z_0. Let Y be an uncountable neighbourhood of z_0 in Z', such that $\delta(Y) < \varepsilon$. Thus, $y, y' \in Y$ implies $\varrho(y, y') < \varepsilon < \varepsilon_y$, i. e. $y' \prec f(y)$, according to (2).

5. **Main theorem.** If $\{T_\iota\}_{\iota\epsilon\Gamma}$ is a collection of τ-sets, we denote by ε_ι, P_ι and Q_ι, respectively, the number and the sets corresponding to T_ι, according to par. 2. Thus, P_ε is a pith of T_ι for every $\iota\epsilon\Gamma$.

THEOREM. *Let $T=\{T_\iota\}_{\iota\epsilon\Gamma}$ be an uncountable collection (i. e. $\varkappa_0<\Gamma$) of τ-sets in locally connected and totally bounded metric space X, such that every two distinct elements of T have disjoint piths.*

Then there exists an uncountable subcollection $T'\subset T$ and a natural order $\prec$ of T' with the property that if $T_\iota, T_\varkappa\epsilon T'$ and $T_\iota\prec T_\varkappa$, then Q_ι transpierces $P_\varkappa$.

Proof. Since Γ is uncountable and $\varepsilon_\varepsilon>0$ for every $\iota\epsilon\Gamma$, there exist an uncountable subset $\Gamma_1\subset\Gamma$ and a number $\varepsilon>0$ such that

(3) $$\varepsilon<\varepsilon_\iota \quad \text{for} \quad \iota\epsilon\Gamma_1.$$

The space X being totally bounded, 2^X is separable ([1], p. 113). We have $\{\overline{P_\iota\cap Q_\iota}\}_{\iota\epsilon\Gamma_1}\subset 2^X$. It follows ([1], p. 140), that the collection $\{\overline{P_\iota\cap Q_\iota}\}_{\iota\epsilon\Gamma_1}$ has a condensation point, $\overline{P_\gamma\cap Q_\gamma}$ $(\gamma\epsilon\Gamma_1)$, Γ_1 being uncountable.

Now, if R_x denotes a connected neighbourhood of the point x in the space X, such that $\delta(R_x)<\varepsilon/2$ (R_x exists by virtue of local connexity of X), then the set

$$R=\bigcup_{x\epsilon\overline{P_\gamma\cap Q_\gamma}} R_x$$

is a region (since $\overline{P_\gamma\cap Q_\gamma}$ is connected) and

$$d(\overline{P_\gamma\cap Q_\gamma}, \overline{R})\leqslant\frac{\varepsilon}{2}.$$

Thus we have $\overline{P_\gamma\cap Q_\gamma}\subset R$. Therefore, there exists a number $\eta>0$ such that if $A\epsilon 2^X$ and $d(A, \overline{P_\gamma\cap Q_\gamma})<\eta$, then $A\subset R$. However, $\overline{P_\gamma\cap Q_\gamma}$ is a condensation point of collection $\{\overline{P_\iota\cap Q_\iota}\}_{\iota\epsilon\Gamma_1}$, then there exist an uncountable subset $\Gamma_2\subset\Gamma_1$ such that the collection $\{\overline{P_\iota\cap Q_\iota}\}_{\iota\epsilon\Gamma_2}$ is a neighbourhood of point $\overline{P_\gamma\cap Q_\gamma}$ and has the diameter less than $\min\{\eta, \varepsilon/2\}$. Thus $\overline{P_\iota\cap Q_\iota}\subset R$ and

$$d(\overline{P_\iota\cap Q_\iota}, \overline{R})\leqslant d(\overline{P_\varepsilon\cap Q_\iota}, \overline{P_\gamma\cap Q_\gamma})+d(\overline{P_\gamma\cap Q_\gamma}, \overline{R})<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon<\varepsilon_\iota$$

for every $\iota\epsilon\Gamma$, according to (3) and (4).

Hence, putting

(5) $$K_\iota=C_{P_\iota\cap Q_\iota}(P_\iota\cap R)_\iota$$

(6) $$L_\iota=C_{P_\iota\cap Q_\iota}(Q_\iota\cap R) \text{ for } \iota\epsilon\Gamma_2,$$

we obtain from the definition (see par. 2, (i) and (ii)) that:

(7) $$K_\iota \text{ separates } R,$$

(8) $$L_\iota\neq P_\iota\cap Q_\iota \text{ for } \iota\epsilon\Gamma_1.$$

We include from (5) that the sets K_ι are for $\iota\epsilon\Gamma_2$ connected and disjoint, because $K_\iota\subset P_\iota$ for $\iota\epsilon\Gamma_2$ and the piths P_ι are disjoint according to the hypothesis. Hence, by virtue of (7), the collection $\{K_\iota\}_{\iota\epsilon\Gamma_2}$ is a non-separated collection of cuttings of R in sense of Whyburn (see [3], p. 42). We include from the theorem of Whyburn ([3], p. 45, (2.2)) that there exists an uncountable subset $\Gamma_3\subset\Gamma_2$ such that K_ι is closed in R and irreducibly separates R into just two components for every $\iota\epsilon\Gamma_3$. The space X being metric and totally bounded, it contains a countable dense subset $\{p_1, p_2, \ldots\}$. Therefore, for every $\iota\epsilon\Gamma_3$ there exists a pair (i_ι, j_ι) of natural numbers, such that the points p_{i_ι} and p_{j_ι} lie in different components of $R-K_\iota$. But Γ_3 is uncountable, hence we can find an uncountable subset $\Gamma_4\subset\Gamma_3$ such that $(i_\iota, j_\iota)=$ constans on Γ_4, i. e. there exists the natural numbers i_0 and j_0 with the property that $i_\iota=i_0$ and $j_\iota=j_0$ for every $\iota\epsilon\Gamma_4$. Put $a=p_{i_0}$ and $b=p_{j_0}$, Hence, K_ι separates R into $C_a(R-K_\iota)$ and $C_b(R-K_\iota)$ for every $\iota\epsilon\Gamma_4$. Since K_ι are connected and disjoint subsets of R, the conditions $\iota\neq\varkappa$ and $\iota,\varkappa\epsilon\Gamma_4$ imply $K_\iota\subset C_a(R-K_\varkappa)$ or $K_\iota\subset C_b(R-K_\varkappa)$.

Now let $\prec$ be an order of the set Γ_4 defined as follows: we write $\iota\prec\varkappa$ provided that $K_\iota\subset C_a(R-K_\varkappa)$ for every distinct elements $\iota,\varkappa$ of Γ_4.

Applying over again the above mentioned theorem of Whyburn, we obtain an uncountable subset $\Gamma_5\subset\Gamma_4$ such that for every $\iota\epsilon\Gamma_5$ the set K_ι has "potential order 2 in R relative to $\{K_\iota\}_{\iota\epsilon\Gamma_5}$", i. e. there exists for every $\iota\epsilon\Gamma_5$ a monotone decreasing sequence of open sets $U_\iota^n\subset R$ $(n=1, 2, \ldots)$ such that

(9) $$K_\iota=\bigcap_{n=1}^{\infty}U_\iota^n \quad \text{for} \quad \iota\epsilon\Gamma_5,$$

and, for every $\iota\epsilon\Gamma_5$, $n=1, 2, \ldots$, the relative boundary $\mathrm{Fr}_R(U_\iota^n)$ is contained in the sum of at most 2 elements of collection $\{K_\iota\}_{\iota\epsilon\Gamma_5}$.

We shall prove that for every $\iota\epsilon\Gamma_5$ there exists a natural number $n(\iota)$ such that

(10) $$L_\iota\cap\mathrm{Fr}_R(U_\lambda^{n(\iota)})\neq 0 \quad \text{for} \quad \iota\epsilon\Gamma_5.$$

Indeed, if not, we have $L_\lambda\cap\mathrm{Fr}_R(U_\lambda^n)=0$ for some $\lambda\epsilon\Gamma_6$ and for every $n=1, 2, \ldots$ However, from (6), (9) and from the inclusion $\Gamma_5\subset\Gamma_2$ we infer that L_λ is a connected subset of R and $0\neq P_\lambda\cap Q_\lambda\subset K_\lambda\cap L_\lambda\subset$ $\subset U_\lambda^n\cap L_\lambda$ for $n=1, 2, \ldots$ Hence, $L_\lambda\subset U_\lambda^n$ for $n=1, 2, \ldots$ It follows, according to (9), that $L_\lambda\subset K_\lambda$ which implies, according to (5), that $L_\lambda\subset P_\lambda$. The formula (6) gives obviously the inclusions $R_\lambda\cap Q_\lambda\subset L_\lambda\subset Q_\lambda$. Thus, $L_\gamma=P_\lambda\cap Q_\lambda$ contrary to (8); and (10) is showed.

Since the boundary $\mathrm{Fr}_R(U_\iota^{n(\iota)})$ is contained in the sum of sets K_ι, where $\iota\epsilon\Gamma_5$, we include from (10) that for every $\iota\epsilon\Gamma_5$ there exists an element $f(\iota)$ of Γ_5, such that

(11) $$L_\iota\cap\mathrm{Fr}_R(U_\iota^{n(\iota)})\cap K_{f(\iota)}\neq 0.$$

It implies that $\iota \neq f(\iota)$, because $K_\iota \cap \mathrm{Fr}_R(U_\iota^{n(\iota)}) = 0$, K_ι being closed, $U_\iota^{n(\iota)}$ — open in R and $K_\iota \subset U_\iota^{n(\iota)}$ for every $\iota \in \Gamma_5$. Hence $\iota \prec f(\iota)$ or $f(\iota) \prec \iota$ for $\iota \in \Gamma_5$. But Γ_5 is uncountable. Therefore there exists an uncountable subset $\Gamma_6 \subset \Gamma_5$ such that for every $\iota \in \Gamma_6$ one and the same of two above relations is true. By virtue of symmetry we can assume that $\iota \prec f(\iota)$ for every $\iota \in \Gamma_6$. Putting

$$\bar{\varrho}(\iota, \varkappa) = d(K_\iota, K_\varkappa) \quad \text{for} \quad \iota, \varkappa \in \Gamma_6,$$

we obtain the metric $\bar{\varrho}$ of the set Γ_6 (because K_ι are closed subsets of R!). It is easy to see, that $\prec$ is a natural order of metric space Γ_6 (cf. [3], p. 41—43).

However, R is a totally bounded set, therefore the space 2^R with the metric d, and also the space Γ_6 with the metric $\bar{\varrho}$, are separable (see [1], p. 113). Thus, applying the lemma of par. 4 for $Z = \Gamma_6$, we obtain an uncountable subset $\Gamma_7 \subset \Gamma_6$ such that

$$(12) \qquad \varkappa \prec f(\iota) \quad \text{for every} \quad \iota, \varkappa \in \Gamma_7.$$

Let $T' = \{T_\iota\}_{\iota \in \Gamma_7}$ and let the order $\prec$ of T' be defined by the order $\prec$ of Γ_7 as follows: $T_\iota \prec T_\varkappa$ if and only if $\iota \prec \varkappa$.

Suppose $T_\iota, T_\varkappa \in T'$ and $T_\iota \prec T_\varkappa$. We infer from (12) that $\iota \prec \varkappa \prec f(\iota)$. That is (compare the definition of $\prec$) the sets K_ι and $K_{f(\iota)}$ lie in different components of $R—K_\varkappa$, namely in $C_a(R — K_\varkappa$ and $C_b(R—K_\varkappa)$ respectively. From (5) we include that K_ι and $K_{f(\iota)}$ lie in different components of $R—P_\varkappa$, because the piths P_ι are disjoint and $K_\varkappa \subset P_\varkappa$. However, (5) and (6) imply $0 \neq P_\iota \cap Q_\iota \subset L_\iota \cap K_\iota$, and (11) implies $L_\iota \cap K_{f(\iota)} \neq 0$. Therefore L_ι intersects both two different components of $R — P_\varkappa$. It means, according to (6), that Q_ι transpierces $P_\varkappa$.

Thus our theorem is showed. It instantly implies

(M*) *Each uncountable collection of τ-sets in locally connected and totally bounded metric space contains at least two elements having a point in common.*

6. **Applications.** Since the plane E^2 can be considered as a totally bounded metric space (for instance by homeomorphical transformation of E^2 onto an interior of disk), the theorem (M) is a consequence of (M^*) (cf. pars. 1, 2 and 5).

However, we shall obtain some other corollaries which are not the consequences of Moore's theorem.

By a *triod* we mean a sum of two arcs $Ł$ and L such that $Ł$ and L have exactly one point in common and this point is an interior point of $Ł$ and an end point of L.

COROLLARY 1. *If $T = \{T_\varepsilon\}_{\iota \in \Gamma}$ is an uncountable collection of triods (where $T_\iota = Ł_\iota \cup L_\iota$ for $\iota \in \Gamma$) lying in the plane and such that $Ł_\iota$ are mutually disjoint, then there exists an uncountable subcollection $T' \subset T$ such that for every $T_\iota, T_\varkappa \in T'$ the arcs L_ι and $Ł_\varkappa$ or $L_\varkappa$ and $Ł_\iota$ are crossed.*

For it is sufficient to compare par. 3 and par. 5, every triod being obviously a τ-set in the plane (with the pith L).

Now let T be a *n-dimensional triod* ($n=1, 2, \ldots$) provided that T is a sum of a n-cell C (i.e. a set homeomorphic with the cube I^n) and an arc L such that C and L have exactly one point in common, being an interior point of C and an end point of L. Hence every triod is a 1-dimensional triod and inversely.

COROLLARY 2. *If $T=\{T_\iota\}_{\iota\in\Gamma}$ is an uncountable collection of n-dimensional triods (where $T_\iota = C_\iota \cup L_\iota$ for $\iota\in\Gamma$) lying in the $(n+1)$-dimensional Euclidean space E^{n+1} ($n=1,2,\ldots$) and such that C_ι are mutually disjoint, then there exists an uncountable subcollection $T'\subset T$ such that for every $T_\iota, T_\varkappa \in T'$ the arc L_ι transpierces $C_\varkappa$ or the arc $L_\varkappa$ transpierces C_ι.*

For it is enough to observe that every n-dimensional triod is a τ-set in E^{n+1} (with the pith C).

A theorem like Corollary 2 was proved by Zarankiewicz in 1934 and, whitout quoting this, by Young jr. in 1944 (see [5] and [4] *). Nevertheless it is called the "Moore-Young Theorem", e.g. in Referativnyĭ Journal (Moscow) 1957, ref. 6917, and in Mathematical Review 1960, ref. 852.

Using the notion of accessibility (see [3], p. 111) we obtain from Corollary 2 the following

COROLLARY 3. *If $\{C_\iota\}_{\iota\in\Gamma}$ is an uncountable collection of disjoint n-cells lying in E^{n+1} ($n=1,2,\ldots$), then there exists an uncountable subcollection $\{C_\iota\}_{\iota\in\Gamma_1}$ (i.e. $\Gamma_1\subset\Gamma$ and $\aleph_0<\overline{\overline{\Gamma_1}}$) such that every interior point of arbitrary n-cell C_ι, where $\iota\in\Gamma_1$, is not accessible from the set*

$$E^{n+1}-\bigcup_{\iota\in\Gamma_1} C_\iota .$$

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*) From [5], hence also from [4], we know only that in each uncountable collection of n-dimensional triods lying in E^{n+1} there are at least two elements having a point in common, i.e. $C_\varepsilon \cap C_\varkappa \neq 0$, or $C_\iota \cap L_\varkappa \neq 0$, or $L_\iota \cap C_\varkappa \neq 0$. or $L_\iota \cap L_\varkappa \neq 0$ for any $\iota\varkappa\in\Gamma$. The Cor. 2 shows that last case in needless.

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MATHEMATICS

On the Common Embedding of Abstract Quasi Algebras into Equationally Definable Classes of Abstract Algebras

by

J. SŁOMIŃSKI

Presented by A. MOSTKOWSKI on March 4, 1960

1. A theory of the common $\mathfrak{B}$-homomorphisms of quasi algebras.

Any sequence

$$\boldsymbol{A} = \langle A, (f_1, A_1), \ldots, (f_\xi, A_\xi), \ldots \rangle_{\xi<\alpha},$$

where A is a non-empty set and, for $\xi < \alpha$, A_ξ is an arbitrary subset of A and f_ξ is a k_ξ-ary function defined on A_ξ with values in A, is called a quasi algebra of the type $\Delta = \langle k_1, \ldots, k_\xi, \ldots \rangle_{\xi<\alpha}$ *). If in the quasi algebra $\boldsymbol{A}$, $A_\xi = A$ for all $\xi < \alpha$, then $\boldsymbol{A}$ is called *an algebra* and is briefly denoted by $\boldsymbol{A} = \langle A, f_1, \ldots, f_\xi, \ldots \rangle_{\xi<\alpha}$. Let

$$\boldsymbol{A} = \langle A, (f_1, A_1), \ldots, (f_\xi, A_\xi), \ldots \rangle_{\xi<\alpha} \quad \text{and} \quad \boldsymbol{A}' = \langle A', (f_1', A_1'), \ldots, (f_\xi', A_\xi'), \ldots \rangle_{\xi<\alpha}$$

be two quasi algebras of the type Δ. A mapping h of A into (onto) A' is called *a homomorphism of quasi algebra* $\boldsymbol{A}$ *into (onto) quasi algebra* $\boldsymbol{A}'$ if for all $\xi < \alpha$ h maps A_ξ into (onto) A_ξ' and for all $a_1, a_2, \ldots, a_{k_\xi}$ in A_ξ we have

$$h(f_\xi(a_1, a_2, \ldots, a_{k_\xi})) = f_\xi'(h(a_1), h(a_2), \ldots, h(a_{k_\xi})).$$

The one-to-one homomorphisms are called *isomorphisms*. Let T be an arbitrary set and let $\{\boldsymbol{A}_t\}_{t\in T}$ be any family of quasi algebras of the type Δ. Moreover, let $\mathfrak{B}$ be an arbitrary class of algebras of the type Δ. If h_t, for $t \in T$, is any homomorphism (isomorphism) of quasi algebra $\boldsymbol{A}_t$ into the same algebra $\boldsymbol{B} \in \mathfrak{B}$, then the family $\boldsymbol{h} = \{h_t\}_{t\in T}$ is called a *common* $\mathfrak{B}$*-homomorphism* ($\mathfrak{B}$*-isomorphism*) *of the quasi algebras* $\boldsymbol{A}_t$, $t \in T$, *into algebra* $\boldsymbol{B}$. If $\boldsymbol{h} = \{h_t\}_{t\in T}$ is a common $\mathfrak{B}$-homomorphism of quasi algebras $\boldsymbol{A}_t$, $t \in T$, into algebra B and the set $\bigcup_{t\in T} h_t(A_t)$ generates the algebra $\boldsymbol{B}$, then we

*) The subset A_ξ may be also empty and then f_ξ is the k_ξ-ary empty function. Hence, it follows that every quasi algebra $\boldsymbol{A} = \langle A, (f_1, A_1), \ldots, (f_n, A_n) \rangle$ of the type $\langle k_1, \ldots, k_n \rangle$ may be also considered as a quasi algebra $\boldsymbol{A} = \langle A, (f_1, A_1), \ldots, (f_n, A_n), (f_{n+1}, \theta), \ldots, (f_{n+m}, \theta) \rangle$ of the type $\langle k_1, \ldots, k_n, k_{n+1}, \ldots, k_{n+m} \rangle$, where θ is the empty subset of A.

say that $\boldsymbol{h}$ *is a common* $\mathfrak{B}$*-homomorphism of quasi algebras* $\boldsymbol{A}_t$, $t \epsilon T$, *almost onto algebra* $\boldsymbol{B}$ and $\boldsymbol{B}$ is also called a *common almost* $\mathfrak{B}$*-homomorphic image of* $\boldsymbol{A}_t$, $t \epsilon T$. An algebra $\boldsymbol{B}$ of type Δ is called a common $\mathfrak{B}$*-extension of the quasi algebras* $\boldsymbol{A}_t$, $t \epsilon T$, if $\boldsymbol{B} \epsilon \mathfrak{B}$ and if there exists a common $\mathfrak{B}$-isomorphism of quasi algebras $\boldsymbol{A}_t$, $t \epsilon T$, into algebra $\boldsymbol{B}$. In the sequel we shall suppose that $\{\boldsymbol{A}_t\}_{t \epsilon T}$ is an arbitrary family of quasi algebras of the type Δ with

$$\boldsymbol{A}_t = \langle A_t, (f_1^t, A_1^t), \ldots, (f_\xi^t, A_\xi^t), \ldots \rangle_{\xi < \alpha}, \quad \text{for} \quad t \epsilon T.$$

It is obvious that there exist such quasi algebras

$$\boldsymbol{A}'_t = A'_t, (f_1'^t, A_1'^t), \ldots, (f_\xi'^t, A_\xi'^t), \ldots \rangle_{\xi < \alpha}, \ t \epsilon T,$$

that: 1^0 for all different t_1 and t_2 in T the sets A'_{t_1} and A'_{t_2} are disjoint and 2^0 for $t \epsilon T$ the quasi algebra $\boldsymbol{A}_t$ is isomorphic to $\boldsymbol{A}'_t$.

Let $\varphi_t(A_t) = A_t$, for $t \epsilon T$, be fixed isomorphism of $\boldsymbol{A}_t$ onto $\boldsymbol{A}'_t$. Moreover, suppose that $\mathfrak{B}$ is an arbitrary equationally definable class of algebras of the type Δ. Therefore, there exist $\mathfrak{B}$-free algebras. Let

$$\boldsymbol{W} = \langle W, F_1, \ldots, F_\xi, \ldots \rangle_{\xi < \alpha}$$

be a $\mathfrak{B}$-free algebra freely generated by the set $\bigcup\limits_{t \epsilon T} A'_t = \bigcup\limits_{t \epsilon T} \varphi_t(A_t)$.

(1.1) Definition. *Let* $\sim_0$ *be the least congruence of the algebra* $\boldsymbol{W}$ *such that for all* $t \epsilon T$ *and all* $\xi < \alpha$ *and all* $a_1, a_2, \ldots, a_{k_\xi}$ *in* A_ξ^t *we have*

$$f_\xi'^t(\varphi_t(a_1), \varphi_t(a_2), \ldots, \varphi_t(a_{k_\xi})) \sim_0 F_\xi(\varphi_t(a_1), \varphi_t(a_2), \ldots, \varphi_t(a_{k_\xi})).$$

The congruences $\sim$ *of* $\boldsymbol{W}$ *with* $\sim_0 \leqslant \sim$ *are called common* $\mathfrak{B}$*-regularizers of the quasi algebras* $\boldsymbol{A}_t$, $t \epsilon T$. *The congruence* $\sim_0$ *is called the minimal common* $\mathfrak{B}$*-regularizer of* $\boldsymbol{A}_t$, $t \epsilon T$.

THEOREM 1. *If* $\sim$ *is a common* $\mathfrak{B}$*-regularizer of quasi algebras* $\boldsymbol{A}_t$, $t \epsilon T$, *then the family* $\boldsymbol{j}_\sim = \{t_\sim^t\}_{t \epsilon T}$, *where for* $t \epsilon T$

$$j_\sim^t(a) = \varphi_t(a)/\sim, \qquad \text{for} \qquad a \epsilon A_t,$$

is a common $\mathfrak{B}$*-homomorphism of quasi algebras* $\boldsymbol{A}_t$, $t \epsilon T$, *almost onto* $\boldsymbol{W}/\sim$ *). *The family* $\boldsymbol{j}_\sim$ *is a common* $\mathfrak{B}$*-isomorphism if and only if the congruence* $\sim$ *has the following property:*

(r_1) *for all* $t \epsilon T$ *and all* a_1, a_2 *in* A_t

$$\varphi_t(a_1) \sim \varphi_t(a_2) \text{ implies } a_1 = a_2.$$

Proof. In fact, for all $t \epsilon T$ and all $a_1, \ldots, a_{k_\xi}$ in A_ξ^t we have

$$\begin{aligned} j_\sim^t(f_\xi^t(a_1, \ldots, a_{k_\xi})) &= \varphi_t(f_\xi^t(a_1, \ldots, a_{k_\xi}))/\sim = \\ &= f_\varepsilon'^t(\varphi_t(a_1), \ldots, \varphi_t(a_{k_\xi}))/\sim = f_\xi'^t/\sim(\varphi_t(a_1)/\sim, \ldots, \varphi_t(a_{k_\xi})/\sim) = \\ &= f_\xi'^t/\sim(j_\sim^t(a_1), \ldots, j_\sim^t(a_{k_\xi})) \end{aligned}$$

*) By $W/\sim$ is denoted the quotient algebra formed by dividing algebra $\boldsymbol{W}$ by congruence $\sim$. By $w/\sim$, for $w \epsilon W$ is denoted the abstraction class of $\sim$ determined by w. For the theory of algebras see [3].

or the family $j_\sim$ is a common $\mathfrak{V}$-homomorphism of A_t, $t\in T$, almost onto $W/_\sim$ (since the set $\bigcup_{t\in T} \varphi_t(A_t)$ generates W). The family $j_\sim$ is a common $\mathfrak{V}$-isomorphism if and only if for $t\in T$ and a_1, a_2 in A_t the condition $\varphi_t(a_1)\sim\varphi_t(a_2)$ implies $\varphi_t(a_1)=\varphi_t(a_2)$ which is equivalent to $a_1=a_2$, because φ_t is one-to-one. Thus, Theorem 1 is proved. The common $\mathfrak{V}$-homomorphism $j_\sim$ defined in Theorem 1 is called *induced by the common $\mathfrak{V}$-regularizer* $\sim$. Let $h=\{h_t\}_{t\in T}$ be an arbitrary common $\mathfrak{V}$-homomorphism of quasi algebras A_t, $t\in T$, into an algebra $B=\langle B, f_1^*, \ldots, f_\xi^*, \ldots\rangle_{\xi<\alpha}$. Since W is a $\mathfrak{V}$-free algebra freely generated by the set $\bigcup_{t\in T}\varphi_t(A_t)$, the mapping $\varphi_t(a)\rightarrow h_t(a)$, for all $t\in T$ and $a\in A_t$, may be extended to a homomorphism u_h of W into B. Let $\sim_h$ be the congruence of W induced by homomorphism u_h (i. e. $w_1\sim_h w_2$ if and only if $u_h(w_1)=u_h(w_2)$). Then we have

THEOREM 2. *The congruence $\sim_h$ is a common $\mathfrak{V}$-regularizer of quasi algebras A_t, $t\in T$.*

Proof. In fact, for all $t\in T$, all $\xi<\alpha$, and all $a_1,\ldots,a_{k_\xi}$ in the set A_ξ^t we have $u_h(f_\xi'^t(\varphi_t(a_1),\ldots,\varphi_t(a_{k_\xi})))=u_h(\varphi_t(f_\xi^t a_1,\ldots,a_{k_\xi}))=h_t(f_\xi^t(a_1,\ldots,a_{k_\xi}))=$ $=f_\xi^*(h_t(a_1),\ldots,h_t(a_{k_\xi}))=f_\xi^*(u_h(\varphi_t(a_1)),\ldots,u_h(\varphi_t(a_{k_\xi})))=u_h(F_\xi(\varphi_t(a_1),\ldots.$ $\varphi_t(a_{k_\xi})))$ or $\sim_0\leqq\sim_h$ and this concludes our proof. The common $\mathfrak{V}$-regularizer $\sim_h$ of A_t, $t\in T$, is called *induced by the common $\mathfrak{V}$-homomorphism* h. Let $h=\{h_t\}_{t\in T}$ be any common $\mathfrak{V}$-homomorphism of quasi algebras A_t, $t\in T$, into an algebra B. Let $\sim_h$ be the common $\mathfrak{V}$-regularizer of A_t, $t\in T$, induced by h. Moreover, let $j_{\sim h}=\{j^t_{\sim h}\}_{t\in T}$ be the common $\mathfrak{V}$-homomorphism of A_t, $t\in T$, induced by $\sim_h$. Obviously the mapping $i_h(w/_{\sim h})=$ $=u_h(w)$, for $w\in W$, is an isomorphism of $W_{/\sim h}$ into B (onto subalgebra B_1 of B generated by the set $\bigcup_{t\in T} h_t(A_t)$). For all $t\in T$ and all $a\in A_t$ we have $i_h\, j^t_{\sim h}\, a)=i_h(\varphi_t(a)_{/\sim h})=u_h(\varphi_t(a))=h_t(a)$ or for all $t\in T$ $h_s=i_h\, j^t_{\sim h}$. Since i_h is one-to-one, it follows from Theorem 1 that h_t is an isomorphism if and only if the condition $\varphi_t(a_1)\sim_h\varphi_t(a_2)$ implies $a_1=a_2$, for a_1, a_2 in A_t. Therefore, we have proved:

THEOREM 3. *Let $h=\{h_t\}_{t\in T}$ be any common $\mathfrak{V}$-homomorphism of quasi algebras A_t, $t\in T$, and let $\sim_h$ be the common $\mathfrak{V}$-regularizer of A_t, $t\in T$, induced by h. Then, for all $t\in T$,*

$$h_t=i_h\, j^t_{\sim h},$$

where $f_{\sim h}=\{j^t_{\sim h}\}_{t\in T}$ is the common $\mathfrak{V}$-homomorphism of A_t, $t\in T$, induced by $\sim_h$. The family h is a common $\mathfrak{V}$-isomorphism of A_t, $t\in T$, if and only if $\sim_h$ has the following property:

(r_2) *for all $t\in T$ all $a_1, a_2\in A_t$*

$$\textit{the condition } \varphi_t(a_1)\sim_h\varphi_t(a_2) \textit{ implies } a_1=a_2.$$

From Theorem 3 results

THEOREM 4. *Let A_t, $t\in T$, be an arbitrary family of quasi algebras of the type Δ and let $\mathfrak{V}$ be any equationally definable class of algebras of the*

type Δ. In order that there exist a common $\mathfrak{B}$-extension of quasi algebras A_t, $t \epsilon T$, it is necessary and sufficient that the following property be satisfied:

(r_3) *for all $t \epsilon T$ and all a_1, a_2 in A_t*

$$\text{the condition } \varphi_t(a_1) \sim_0 \varphi_t(a_2) \text{ implies } a_1 = a_2,$$

where $\sim_0$ is the minimal common $\mathfrak{B}$-regularizer of A_t, $t \epsilon T$.

Proof. In fact, if $\boldsymbol{B}$ is a common $\mathfrak{B}$-extension of A_t, $t \epsilon T$, then there exists a common $\mathfrak{B}$-isomorphism $\boldsymbol{h} = \{h_t\}_{t \epsilon T}$ of A_t, $t \epsilon T$, into algebra $\boldsymbol{B}$. By Theorem 3 the property (r_2) holds. Since $\sim_0 \leqslant \sim_{\boldsymbol{h}}$, then from (r_2) follows (r_3). Conversely, if the property (r_3) holds, then by Theorem 1 the common $\mathfrak{B}$-homomorphism $\boldsymbol{j}_{\sim_0} = \{j^t_{\sim_0}\}_{t \epsilon T}$ induced by $\sim_0$ is a common $\mathfrak{B}$-isomorphism of A_t, $t \epsilon T$, almost onto $\boldsymbol{W}/_{\sim_0}$ or $\boldsymbol{W}/_{\sim_0}$ is a common $\mathfrak{B}$-extension of A_t, $t \epsilon T$, and Theorem 4 is proved.

Let $\boldsymbol{g}_1 = \{g_1^t\}_{t \epsilon T}$ and $\boldsymbol{g}_2 = \{g_2^t\}_{t \epsilon S}$ be two common $\mathfrak{B}$-homomorphisms of quasi algebras A_t, $t \epsilon T$, first almost onto an algebra $\boldsymbol{B}_1$, second into algebra $\boldsymbol{B}_2$. Let $\sim_{\boldsymbol{g}_1}$ and $\sim_{\boldsymbol{g}_2}$ be the common $\mathfrak{B}$-regularizers induced by $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$. It is easy to prove (see proof of Theorem 6 in [2]) the next

THEOREM 5. *In order that $\sim_{\boldsymbol{g}_1} \leqslant \sim_{\boldsymbol{g}_2}$ it is necessary and sufficient that there exists exactly one homomorphism p of algebra $\boldsymbol{B}_1$ into algebra $\boldsymbol{B}_2$ such that for $t \epsilon T$*

$$g_2^t = p g_1^t.$$

The homomorphism p maps "onto" if and only if $\boldsymbol{g}_2$ is a common $\mathfrak{B}$-homomorphism of A_t, $t \epsilon T$, almost onto $\boldsymbol{B}_2$.

From Theorem 1 and from the proof of Theorem 3 it follows that an algebra $\boldsymbol{B} \epsilon \mathfrak{B}$ is a common almost $\mathfrak{B}$-homomorphic image of quasi algebras A_t, $t \epsilon T$ if and only if it is isomorphic to an algebra of the form $\boldsymbol{W}/_{\sim}$, where $\sim$ is a common $\mathfrak{B}$-regularizer of A_t, $t \epsilon T$.

If we suppose that the set T has only one element, then from Theorems 1, 2, 3, 4 and 5 result the Theorems 1, 2, 3, 4 and 6 of [2].

2. $\mathfrak{B}$-free products of quasi algebras

A definition of a *$\mathfrak{B}$-free product of algebras* was given by R. Sikorski [1]. The $\mathfrak{B}$-free product (where $\mathfrak{B}$ is an equationally definable class), in the sense of Sikorski, of algebras A_t, $t \epsilon T$, does not always exist. Now we shall give a new definition of a $\mathfrak{B}$-free product: there always exists a $\mathfrak{B}$-free product of A_t, $t \epsilon T$, where A_t, $t \epsilon T$ can be not only algebras but also quasi algebras. In the following we shall introduce the notion of *proper $\mathfrak{B}$-free product* which coincides (in the case of algebras) with the $\mathfrak{B}$-free product in the sense of Sikorski.

Let A_t, $t \epsilon T$, be any family of quasi algebras of the type Δ and let $\mathfrak{B}$ be an arbitrary class of algebras of the type Δ.

(2.1) *An algebra* $\boldsymbol{B}$ *of the type* Δ *is called* $\mathfrak{V}$*-free product of quasi algebras* $\boldsymbol{A}_t$, $t \epsilon T$, *if it has the following properties:*

(2.1.a) $\boldsymbol{B} \epsilon \mathfrak{V}$.

(2.1.b) *there exists a common* $\mathfrak{V}$*-homomorphism* $\boldsymbol{g} = \{g_t\}_{t \epsilon T}$ *of quasi algebras* $\boldsymbol{A}_t$, $t \epsilon T$, *almost onto algebra* $\boldsymbol{B}$;

(2.1.c) *for every common* $\mathfrak{V}$*-homomorphism* $\boldsymbol{h} = \{h_t\}_{t \epsilon T}$ *of quasi algebras* $\boldsymbol{A}_t$, $t \epsilon T$, *into an arbitrary algebra* $\boldsymbol{C} \epsilon \mathfrak{V}$, *the mapping* $g_t(a) \rightarrow h_t(a)$, *for all* $t \epsilon T$ *and all* $a \epsilon A_t$, *may be extended to a homomorphism of the algebra* $\boldsymbol{B}$ *into algebra* $\boldsymbol{C}$.

It may be easily proved that

(2.2) *If* $\boldsymbol{B}$ *and* $\boldsymbol{B}'$ *are two* $\mathfrak{V}$*-free products of quasi algebras* $\boldsymbol{A}_t$, $t \epsilon T$, *then the algebras* $\boldsymbol{B}$ *and* $\boldsymbol{B}'$ *are isomorphic.*

Hence, it follows that the $\mathfrak{V}$-free product of $\boldsymbol{A}_t$, $t \epsilon T$, is (if it exists) uniquely determined up to isomorphisms by $\boldsymbol{A}_t$, $t \epsilon T$. Let (2.1.b′) denote the following property:

There exists a common $\mathfrak{V}$*-isomorphic* $\boldsymbol{g} = \{y_t\}_{t \epsilon T}$ *of quasi algebras* $\boldsymbol{A}_t$, $t \epsilon T$, *almost onto algebra* $\boldsymbol{B}$.

Now we give the definition of the proper $\mathfrak{V}$-free product.

(2.3) *An algebra* $\boldsymbol{B}$ *is called a proper* $\mathfrak{V}$*-free product of quasi algebras* $\boldsymbol{A}_t$, $t \epsilon T$, *if it has the properties* (2.1.a), (2.1.b′) and (2.1.c).

Obviously a proper $\mathfrak{V}$-free product of quasi algebras $\boldsymbol{A}_t$, $t \epsilon T$, is a common $\mathfrak{V}$-extension of $\boldsymbol{A}_i$, $t \epsilon T$, and is (if it exists) uniquely determined up to isomorphism by $\boldsymbol{A}_t$, $t \epsilon T$.

Now we shall prove that, if $\mathfrak{V}$ is an equationally definable class of algebras, then there exists always a $\mathfrak{V}$-free product of quasi algebras.

THEOREM 6. *Let* $\mathfrak{V}$ *be any equationally definable class of algebras of the type* Δ *and let* $\boldsymbol{A}_t$, $t \epsilon T$, *be an arbitrary family of quasi algebras of the type* Δ. *Moreover, let* $\sim_0$ *be the minimal common* $\mathfrak{V}$*-regularizer of quasi algebras* $\boldsymbol{A}_t$, $t \epsilon T$. *Then the algebra* $\boldsymbol{W}/_{\sim_0}$ *is the* $\mathfrak{V}$*-free product of quasi algebras* $\boldsymbol{A}_t$, $t \epsilon T$.

P r o o f. By Theorem 1 the family $\boldsymbol{j}_{\sim_0} = \{j^t_{\sim_0}\}_{t \epsilon T}$ is a common $\mathfrak{V}$-homomorphism of $\boldsymbol{A}_t$, $t \epsilon T$, almost onto $\boldsymbol{W}/_{\sim_0}$. Let $\boldsymbol{h} = \{h_t\}_{t \epsilon T}$ be any common $\mathfrak{V}$-homomorphism of $\boldsymbol{A}_t$, $t \epsilon T$, into an arbitrary algebra $\boldsymbol{C} \epsilon \mathfrak{V}$. Let $\sim_{\boldsymbol{h}}$ be the common $\mathfrak{V}$-regularizer of $\boldsymbol{A}_t$, $t \epsilon T$, induced by $\boldsymbol{h}$ and let $\boldsymbol{j}_{\sim \boldsymbol{h}} = \{j^t_{\sim \boldsymbol{h}}\}_{t \epsilon T}$ be the common $\mathfrak{V}$-homomorphism of $\boldsymbol{A}_t$, $t \epsilon T$, induced by $\sim_{\boldsymbol{h}}$. By Theorem 3 for all $t \epsilon T$, $h_t = i_{\boldsymbol{h}}\, j^t_{\sim \boldsymbol{h}}$. Since $\sim_0 \leqslant \sim_{\boldsymbol{h}}$, the mapping $q(w/_{\sim_0}) = w/_{\sim \boldsymbol{h}}$, for w in W, is a homomorphism of $\boldsymbol{W}/_{\sim_0}$ onto $\boldsymbol{W}/_{\sim \boldsymbol{h}}$. The function $p = i_{\boldsymbol{h}}\, q$ is a homomorphism of $\boldsymbol{W}/_{\sim \boldsymbol{h}}$ into $\boldsymbol{C}$ such that for all $t \epsilon T$ and all $a \epsilon A_t$ we have $p(j^t_{\sim_0}(a)) = p(\varphi_t(a)/_{\sim_0}) = i_{\boldsymbol{h}}(\varphi_t(a)/_{\sim \boldsymbol{h}}) = i_{\boldsymbol{h}}(j^t_{\sim \boldsymbol{h}}(a)) = i_{\boldsymbol{h}}\, j^t_{\sim \boldsymbol{h}}(a) = h_t(a)$ or $\boldsymbol{W}/_{\sim_0}$ is the $\mathfrak{V}$-free product of $\boldsymbol{A}_t$, $t \epsilon T$ and the Theorem 6 is proved.

In the sequel we shall suppose that $\mathfrak{V}$ is any equationally definable class of algebras of the type Δ.

THEOREM 7. *If A_t, $t \epsilon T$, is such a family of quasi algebras of the type Δ that there exists a common $\mathfrak{B}$-extension of A_t, $t \epsilon T$, then there also exists the proper $\mathfrak{B}$-free product of A_t, $t \epsilon T$.*

Proof. By Theorems 4 and 1 the family $j_{\sim_0} = \{j^t_{\sim_0}\}_{t \epsilon T}$ is a common $\mathfrak{B}$-isomorphism of A_t, $t \epsilon T$. Thus, from the proof of Theorem 6 it follows that $W/_{\sim_0}$ is the proper $\mathfrak{B}$-free product of quasi algebras A_n, $t \epsilon T$, and Theorem 7 is proved. If we suppose in Theorem 7 that all A_t, $t \epsilon T$, are algebras belonging to class $\mathfrak{B}$, then from Theorem 7 we obtain Theorem contained in Sikorski's paper [1]. The next theorem gives the necessary and sufficient condition for the existence of a proper $\mathfrak{B}$-product of quasi algebras.

THEOREM 8. *Let A_t, $t \epsilon T$, be any family of quasi algebras of the type Δ and let $\mathfrak{B}$ be an arbitrary equationally definable class of algebras of the type Δ. Moreover, let $\sim_0$ be the minimal common $\mathfrak{B}$-regularizer of A_t, $t \epsilon T$. Then there exists the proper $\mathfrak{B}$-free product of quasi algebras A_t, $t \epsilon T$, if and only if the relation $\sim_0$ has the following property:*

(r_3) *for all $t \epsilon T$ and all a_1, a_2 in A_t*

$$\text{the condition } \varphi_t(a_1) \sim_0 \varphi_t(a_2) \text{ implies } a_1 = a_2.$$

Proof. If the proper $\mathfrak{B}$-free product of A_t, $t \epsilon T$, exists, then it is a common $\mathfrak{B}$-extension of A_t, $t \epsilon T$, and therefore, by Theorem 4 the property (r_3) holds. Conversely, if the property (r_3) holds, then by Theorem 4 there exists a common $\mathfrak{B}$-extension of A_t, $t \epsilon T$, and thus, by the Theorem 7, there exists the proper $\mathfrak{B}$-free product of A_t, $t \epsilon T$, which is the algebra $W/_{\sim_0}$. This concludes our proof.

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MATHEMATICS

Number of Algebras with a Given Set of Elements

by

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Given a set S we consider the family of all algebras (see [1], p. vii) for which S is the set of elements. E. L. Post [2] proved that if S has two elements, the number of non-isomorphic algebras on S is then $\aleph_0$. In this note we give the cardinals of families of non-isomorphic algebras defined on sets S with more than two elements. We prove the following

THEOREM. *If S is a finite set of more than two elements or $\overline{S} = \mathfrak{m}$, $\mathfrak{m} \geqslant \aleph_0$ the number of non-isomorphic algebras on S is then an equal continuum, or $2^{2^{\mathfrak{m}}}$, respectively.*

P r o o f. It is sufficient to prove the Theorem by replacing "non-isomorphic" with "different". Supposing indeed we have proved it with this alternation, we can decompose the class of different algebras on S into subclasses of isomorphic algebras. The cardinals of the subclasses do not exceed the number of all permutations of S which in each case is smaller than the cardinal of the class of all algebras. Hence, selecting from each class one algebra, we obtain the result.

It is sufficient also to prove Theorem with "at least" instead of "equal". To see this, note that the number of algebras on S does not exceed $2^{\aleph_0}$ or $2^{2^{\mathfrak{m}}}$, respectively, because it cannot be greater than the number of all subclasses of the class of all possible operations on S.

To show that there are at least continuum algebras on a set $S = \{0, 1, 2, ..., r\}$, where $r \geqslant 2$, we construct a sequence of operations

$$(1) \qquad (x_1, x_2)_2, \ (x_1, x_2, x_3)_3, ..., \ (x_1, ..., x_k)_k, ...,$$

on S with the property that any operation $(x_1, ..., x_k)_k$ is not in the algebra defined by other operations $(x_1, ..., x_n)_n$, $n \neq k$. If σ is a subsequence of the sequence $\{2, 3, 4, ...\}$ we denote by A_σ the algebra with operations $(x_1, ..., x_n)_n$, where $n \in \sigma$. All algebras A_σ are then different and thus we have continuum algebras on S.

Put for $k \geqslant 2$

$$(x_1, \ldots, x_k)_k = \begin{cases} 0 & \text{if} \quad x_i, x_j \quad \text{for some} \quad i \neq j, \\ 1 & \text{otherwise.} \end{cases}$$

Let σ be any subsequence of the sequence $\{2, 3, 4, \ldots\}$. We have to show that if $k \notin \sigma$, then $(x_1, \ldots, x_k)_k$ is not an operation in A_σ. Assuming the converse, we have, by the symmetry of any operation $(x_1, \ldots, x_m)_m$, an identical equality:

$$(x_1, \ldots, x_k)_k \equiv (x_{i_1}, \ldots, x_{i_2}, x_{i_r}, f_{r+1}, \ldots, f_{r+s})_{r+s}, \tag{2}$$

where the indices $i_1, \ldots, i_r \in \{1, \ldots, k\}$ are not necessarily all different, $f_{r+1}, \ldots, f_{r+s}$ are non-identity operations in the algebra A_σ and $r+s \in \sigma$. This, as we shall show, leads to contradiction.

First, observe that $f_{r+j} \leqslant 1$, $j = 1, 2, \ldots, s$. If we now substitute in (2): $x_i = 2$, $i = 1, \ldots, k$, then the left hand side is equal to 1. If $s \geqslant 2$, then we have 0 on the right. Hence, $s \leqslant 1$. Let us assume first that $s = 1$. Let $i_1 = t \in \{1, \ldots, k\}$. We substitute $x_t = 1$ and we write 2 for any other variable x_j. Then we have, in (2), the value 1 on the left and on the right, by $f_{r+1} \leqslant 1$, we have 0. Finally, assume $s = 0$, i.e. $(x_1, \ldots, x_k)_k \equiv (x_{i_1}, \ldots, x_{i_r})_r$. If $r > k$, then we have $i_j = i_m$ for some $j \neq m$ and a contradiction is obtained by putting 1 for the variable x_n with $n = i_j = i_m$ and writing 2 for all others. The inequality $r < k$ is also impossible since $(x_1, \ldots, x_k)_k$ evidently depends on every variable. Therefore, (2) does not hold.

To show that there are at least $2^{2^{\mathfrak{m}}}$ algebras on a set S with $\mathfrak{m}$, $\mathfrak{m} \geqslant \aleph_0$ elements it is enough to have a family F of $2^{\mathfrak{m}}$ operations on S which has the same property as the sequence (1). We obtain an example of such a family F if we pick out two elements $0, 1 \in S$ and define for each set $B \subset S \setminus \{0, 1\}$ an operation f_B of one variable by the conditions: $f_B(0) = 0$, $f_B(1) = 1$, and, if $x \neq 0, 1$ then $f_B(x) = 1$ or 0, depending whether x belongs to B or not.

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MATHEMATICS

Absolute-Valued Algebras

by

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An algebra A over the real field R is a vector space over R which is closed with respect to a product xy which is linear in both x and y, and which satisfies the condition $\lambda(xy)=(\lambda x)y=x(\lambda y)$ for any λ in R and x, y in A. The product is not necessarily associative. An element e of the algebra A is called a unit element, if $ex=xe=x$ for any x in A. An algebra over R is called *absolute-valued,* if it is a normed space under a multiplicative norm $|\ |$, i.e. a norm satisfying, in addition to the usual requirements, the condition $|xy|=|x|\,|y|$ for any x, y in A. It is obvious that an absolute-valued algebra contains no divisors of zero.

The following theorem gives the complete representation of absolute-valued algebras with a unit element.

THEOREM 1. *An absolute-valued algebra with a unit element is isomorphic to either the real field, the complex field, the quaternion algebra, or the Cayley-Dickson algebra.*

A. A. Albert had previously established this result under the restriction that the algebra is algebraic, in the sense that every element generates a finite dimensional subalgebra [1], [2], [3]. F. B. Wright had proved [5] the same theorem for absolute-valued division algebras with a unit element.

A simple example shows that the assumption of the presence of a unit element is essential. Let A_0 be the space of all sequence $x=\{x_n\}$ of real numbers with convergent series $\sum_{n=1}^{\infty} x_n^2$. A_0 is a Hilbert space over R with respect to the norm $|x|=\left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2}$, and with the usual addition and scalar multiplication $\{x_n\}+\{y_n\}=\{x_n+y_n\}$, $\lambda\{x_n\}=\{\lambda x_n\}$. Let φ be a one-to-one correspondence of the set of all pairs of natural numbers onto the set of all natural numbers. We define the multiplication of elements in A_0 as follows: $\{x_n\}\{y_n\}=\{z_n\}$, where $z_{\varphi(m,k)}=x_m y_k$ $(m, k=$

$=1,2,\ldots$). This product makes A_0 an algebra over R. Moreover, A_0 is absolute-valued. Indeed, we have the equality

$$|x|=\left(\sum_{n=1}^{\infty} z_n^2\right)^{1/2}=\left(\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} z_{\varphi(m,k)}^2\right)^{1/2}=$$
$$=\left(\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} x_m^2\, y_k^2\right)^{1/2}=\left(\sum_{m=1}^{\infty} x_m^2\right)^{1/2}\left(\sum_{k=1}^{\infty} y_k^2\right)^{1/2}=|x|\,|y|.$$

Since A_0 is a Hilbert space, the function $|x|^2$ from A_0 to R is a quadratic form admitting composition with respect to this multiplication. The complete structure theory of algebras (over any field) admitting such a form has been given by I. Kaplansky [4], under the hypothesis that the algebra has a unit element. The infinite dimensional algebra A_0 above shows that this hypothesis of existence of a unit element is essential to Kaplansky's results.

THEOREM 2. *If an absolute-valued algebra A contains an element $a\neq 0$ which commutes with every element of A and which is alternative, i.e. which satisfies the equations $a(ax)=a^2x$, $(xa)a=xa^2$ for any x in A, then A has a unit element.*

THEOREM 3. *A commutative absolute-valued algebra is isomorphic to either the real field, the complex field, or the algebra of complex numbers with the product of x and y defined to be $\bar{x}\bar{y}$.*

The proof will be published in the Proceedings of the American Mathematical Society.

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MATHEMATICS

Non-uniqueness Results for Transfinite Progressions

by

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Presented by A. MOSTOWSKI on March 13, 1960

1. Suppose sets A_y of natural numbers or sets C_y of functions of natural numbers are associated with ordinal notations y of a subclass O^* of O. We assume throughout $(z \epsilon O^* \,\&\, y <_0 z) \rightarrow y \epsilon O^*$. The association is called *progressive* if, for $y <_0 z$, $A_y \subset_+ A_z$, and $C_y \subset_+ C_z$ respectively. ($\subset_+$ means: is properly contained in), and *recursive*, if there are (primitive) recursive relations R and S such that, for $y \epsilon O^*$

$$A_y = \{n \,|\, (Et)\, R(y; n; t)\},$$

$$C_y = \{E_p \,|\, F_p = \{\langle n, m\rangle \,|\, S(y; p, n, m)\}\}\ ^{*)}.$$

PROBLEM. *For what ordinals α do we get a hierarchy?, i.e. if the collections* $\{A_y\}$ and $\{C_y\}$ *are recursive and progressive, for what α and β do we get:*

$$(y \epsilon O^* \,\&\, z \epsilon O^* \,\&\, |y| = |z| \leqslant \alpha) \rightarrow A_y = A_z$$

$$(y \epsilon O^* \,\&\, z \epsilon O^* \,\&\, |y| = |z| \leqslant \beta) \rightarrow C_y = C_z.$$

We shall show that, under very simple conditions on O^*, which merely require that O^* contains notations for certain well-orderings described below, we have $\alpha \leqslant \omega^3$ and $\beta \leqslant \omega^4$.

D i s c u s s i o n. The theorem may be applied to Turing's ordinal logics which have been further examined by Feferman, since there sets of numbers A_y (Gödel numbers of theorems in the logic associated with y) which are both recursive and progressive are considered. Here O^* is O itself. Another application is to Kleene's progression of classes of number theoretic functions in [1], where O^* consists of his *primitive* recursive ordinal notation. In [2], one considers both sets of theorems and sets of functions, where now O^* is restricted to *provable* notations, i.e. numbers n,

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*) The p^{th} function in C_y is identified with its graph F_y and so $S(y; p, n, m)$ means that the value of the p^{th} function in C_y at n is m. We require, when $y \epsilon 0^*$, $(p)\,(n)\,(E; m)\, S(y; p, n, m)$.

where, for a suitable formalization of membership in O, one can prove $0^{(n)} \epsilon O$ by restricted means. We establish our result for weak conditions on O^* in order to have this wide range of application. It is to be emphasized that the generality consists in allowing O^* to be so small that every n in O^* is easily recognized to be an ordinal notation, while in general, for $n \epsilon O^*$, $m \epsilon O^*$ it will be difficult to decide whether $|n|=|m|$. In fact, in most of the applications, if $|n|=|m|$ is *provable* by suitably restricted means, we shall have $A_n = A_m$ and $C_n = C_m$.

We do not know whether the bounds on α and β are optimal, i.e. minimal, under our general conditions. They are certainly not optimal in special cases, e.g. for Kleene's classes of number theoretic functions as shown by Axt in his solution of problem P238 in [1].

The method of the present paper is closely related to [3] and [4].

2. We call $\{A_y\}$, $\{C_y\}$ α-unique (for notations y and z in O^*), if

$$|y|=|z| \leqslant \alpha \rightarrow A_y = A_z, \quad \text{resp.} \quad |y|=|z| \leqslant \alpha \rightarrow C_y = C_z.$$

LEMMA 1. *If $\{A_y\}$ is progressive and α-unique, then for $|y| \leqslant \alpha$, $|z| \leqslant \alpha$,*

$$|y|=|z| \leftrightarrow A_y = A_z.$$

$\rightarrow$ is state explicitly in the hypothesis. For the converse we only require min $(|y|, |z|)$-uniqueness. For, suppose $|y| \neq |z|$, say $|y| < |z|$ and $t <_0 z$ and $|t|=|y|$. By definition of O^*, $z \epsilon O^* \rightarrow t \epsilon O^*$.

By progressiveness $A_t \subset_+ A_z$, by assumed uniqueness $A_t = A_y$, hence $A_y \neq A_z$.

Similarly for $\{C_y\}$. Note that recursiveness was not needed.

LEMMA 2. *For $y \epsilon O^*$ and $z \leqslant O^*$.*

(i) *if $\{A_y\}$ is recursive, $A_y = A_z \leftrightarrow (u)(Ev)E(u,v)$,*

(ii) *if $\{C_y\}$ is recursive, $C_y = C_z \leftrightarrow (u)(Ev)(w)I(u,v,w)$*

with primitive recursive E and I.

(i) $A_y = A_z : (n)(x)(Es)(Et)\{[R(y,n,x) \rightarrow R(z,n,s)]\,[R(z,n,x) \rightarrow R(y,n,t)]\}$.

(ii) $C_y = C_z : (p)(Eq)(Eq_1)(n)(m)\{[S(y;p,m,n) \rightarrow S(z;q,m,n)]\,[S(z;p,m,n) \rightarrow S(y;q_1,m,n)]\}$.

The usual contraction does the rest.

Finally, we show how every Σ_2^O predicate can be reduced to $|y| = \omega^3$ and Σ_3^O to $|y| = \omega^4$, for y in O^* ***provided only that O^* contains the simple notations obtained by the U-procedure and E-procedure below.***

(a) ***U-procedure*** (universal formulae). Given $(y) R(a\ y)$ let

$$f_a(0)=1,\quad f_a(m+1)=f_a(m)+_O \begin{cases} w & \text{if } (y)_{\leqslant m} R(n,y) \\ 2 & \text{if } (Ey)_{\leqslant m} > R(n,y), \end{cases}$$

where 2 is the notation for unity and w a notation for ω in O^*.

The procedure is uniform in a. We assume that $3_O\, 5^{e_f}$ is in O^* if e_f is the number of the recursion equation above defining f_a. If $(y) R(n,y)$, we have a notation for ω^2; if $> (y) R(n,y)$, we have one for $w.\ k$.

To reduce $(Ex)(y) R(x,y)$, we rewrite $(Ex)_{\leqslant n}(y) R(x,y)$ in the usual way, as $(y) R'(n,y)$ and let f'_n be the notation associated with $(y)R'(n,y)$ by the above procedure.

$$g(0)=1,\quad g(n+1)=g(n)+_O f'_n.$$

g yields a notation for ω^3, if $(Ex)(y)R(x,y)$ is true and for ω^2, if false.

Hence, provided notations obtained by the U-procedure are contained in O^*, we have reduced Σ_2^O to

$$|n|=\omega^3.$$

Note that the relevant values of n automatically satisfy $|n|\leqslant\omega^3$. In particular, we do not require notations in O^* of ordinals $>\omega^3$ at all.

(b) ***E-procedure.*** Given $(Ez) R(a,b,z)$, let

$$f_{a,b}(0)=1,\qquad f_{a,b}(n+1)=f_{a,b}(n)+_O \begin{cases} w & \text{if } (Ez)_{\leqslant n+1} R(a,b,z) \\ 2 & \text{if } (z)_{\leqslant n+1} \to R(a,b,z) \end{cases}$$

$f_{a,b}$ yields a notation for ω^2, if $(Ez) R(a,b\ z)$ is true; if it is false, $|f_{a,b}|=\omega.\ k$, for some finite k.

To reduce $(y)(Ez)R(a,y,z)$, we rewrite $(y)_{\leqslant n}(Ez)R(a,y,z)$ as $(Ez) R'(a,n,z)$ and define

$$g_a(0)=1\qquad g_a(n+1)=g_a(n)+_O f'_{a,n}$$

g_a *is* a notation for ω^3 if $(y)(Ez) R(a,y,z)$, for $\omega^2.\ k$ otherwise. Finally, we rewrite $(Ex)_{\leqslant n}(y)(Ez) R(x,y,z)$ as $(y)(Ez) R^*(n,y,z)$ and let g_n^* be the corresponding notation. Then

$$h(0)=1\quad h(n+1)=h(n)+_O g_n^*$$

yields a notation for ω^4 if $(Ex)(y)(Ez) R(x,y,z)$ is true, for $\omega^3\cdot k$ otherwise.

Under the condition that O^* contains the notations employed under (a) and (b) we have

THEOREM (i). *If $\{A_y\}$ is progressive and recursive and α-unique, then $\alpha\leqslant\omega^3$.*

(ii) *If $\{C_y\}$ is progressive, recursive and β-unique, then $\beta\leqslant\omega^4$.*

Proof. Let w_3 and w_4 be notations in O^* for ω^3 and ω^4.

(Such notations exist, because they can be obtained by the above procedures).

(i) By Lemma 1, for $a > \omega^3$, for $|n| \leqslant \omega^3$, $|n| = \omega^3 \leftrightarrow A_n = A_n = A_{w_3}$.

By Lemma 2, $A_n = A_{w_3}$ is Π_2^O. By (a) above Σ_2^O is reducible to $|n| = \omega^3$ with n in O^* and $|n| \leqslant \omega^3$.

This contradicts the hierarchy theorem.

(ii) Same argument as in (i), *mutatis mutandis*.

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MATHEMATICS

Abstract Algebras in which all Elements are Independent

by

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1. Introduction. We adopt the terminology and notation of [1], [2] and [4]. In particular for any algebra $\mathfrak{A}=(A; \boldsymbol{F})$ the class of all algebraic operations in $\mathfrak{A}$ will be denoted by $\boldsymbol{A}(\mathfrak{A})$, and the class of all operations of n variables belonging to $\boldsymbol{A}(\mathfrak{A})$ by $\boldsymbol{A}^{(n)}(\mathfrak{A})$.

Two algebras $\mathfrak{A}=(A; \boldsymbol{F})$ and $\mathfrak{A}^*=(A; \boldsymbol{F}^*)$ are considered as identical, if the classes $\boldsymbol{A}(\mathfrak{A})$ and $\boldsymbol{A}(\mathfrak{A}^*)$ are equal. If $\boldsymbol{A}(\mathfrak{A})\supset\boldsymbol{A}(\mathfrak{A}^*)$, we say that $\mathfrak{A}^*$ is a *subsystem* of $\mathfrak{A}$.

An algebra $\mathfrak{A}$ is called *trivial,* if the class $\boldsymbol{A}(\mathfrak{A})$ contains only trivial operations, i. e. the functions of the form

$$e_j^{(n)}(x_1, \ldots, x_n)=x_j \quad \text{for} \quad x_1, \ldots, x_n \in A \quad j=1, 2, \ldots, n.$$

Let us consider an algebra $\mathfrak{A}=(A; \boldsymbol{F})$ and an n-element subset $I=\{a_1, \ldots, a_n\}$ of A. We say that I is a set of *independent* elements, or else, that all elements of I are independent, if for any $f, g \in \boldsymbol{A}^{(n)}(\mathfrak{A})$ the equality $f(a_1, \ldots, a_n)=g(a_1, \ldots, a_n)$ implies the identity of f and g in A.

All elements of a trivial algebra are obviously independent. S. Świerczkowski proved that, for algebras having at least 3 elements, the converse implication is true: if all elements are independent, then the algebra is trivial ([4], p. 501, and [5], Theorem 1). In order to discuss the remaining case of two-element algebras, let us denote by $\mathcal{J}$ the class of all non-trivial algebras possessing only elements 0 and 1, in which both elements are independent. Using some results of E. L. Post [3], we will prove that *the class $\mathcal{J}$ consists of three algebras,* which will be defined below.

2. Lemma. *Let $\mathfrak{A}(A; \boldsymbol{F})$ be an n-element algebra,* e. g. $A=\{1, 2, \ldots, n\}$. *All elements of A are independent if and only if every operation $f\in\boldsymbol{A}^{(n)}(\mathfrak{A})$ is trivial.*

Proof. Let us suppose that all elements of A are independent and $f\in\boldsymbol{A}^{(n)}(\mathfrak{A})$. We define a trivial operation g of n variables putting for all systems $(x_1, \ldots, x_n)$ of elements of A:

$$g(x_1, x_2, \ldots, x_n)=x_j \quad \text{if} \quad f(1, 2, \ldots, n)=j.$$

Consequently,

$$g(1, 2, .., n) = j = f(1, 2, ..., n)$$

and, since $1, 2, ..., n$ are independent, $f = g$. Thus, f is trivial.

The converse implication is obvious.

3. Definitions. Let us consider the two-element set $T = \{0, 1\}$. We define two T-valued operations p_* and p^* of three variables running over T by the following conditions:

$$(*) \qquad \begin{cases} p_*(x, x, y) = p_*(x, y, x) = p_*(y; x, x) = y, \\ p^*(x, x, y) = p^*(x, y, x) = p^*(y, x, x) = x. \end{cases}$$

In other terms:

$$p_*(x, y, z) = x + y + z \,(\mathrm{mod}\ 2),$$

$$p^*(x, y, z) = xy + yz + xz \,(\mathrm{mod}\ 2).$$

The three algebras:

$$\mathfrak{P}_* = (T; p_*) \qquad \mathfrak{P}^* = (T; p^*) \qquad \mathfrak{P} = (T; p_*, p^*)$$

were considered by Post [3].

4. Theorem. $\mathcal{J} = \{\mathfrak{P}_*, \mathfrak{P}^*, \mathfrak{P}\}$.

P r o o f. 1. It follows directly from $(*)$ and from the definition of algebraic operations of two variables (see [1], p. 731), that all these operations in the three considered algebras are trivial. Thus, by Lemma, $\mathfrak{P}_*, \mathfrak{P}^*, \mathfrak{P} \in \mathcal{J}$.

2. In order to prove that $\mathcal{J}$ contains only three considered algebras, we use two following theorems [3], p. 72 and 74):

(a) A T-valued operation f of variables running over $\mathcal{J}$ belongs to $A(\mathfrak{P})$ if and only if it satisfies the following conditions

$$(1) \qquad f(x_1', ..., x_n') = f(x_1, ..., x_n)' \quad \text{where} \quad 0' = 1 \text{ and } 1' = 0$$

$$(2) \qquad f(0, ..., 0) = 0.$$

(b) There exist only two non trivial algebras which are subsystems of $\mathfrak{P}$, namely $\mathfrak{P}_*$ and $\mathfrak{P}^*$.

Consequently, it suffices to prove, that if $\mathfrak{A} \in \mathcal{J}$ and $f \in A^{(n)}(\mathfrak{A})$, then f satisfies (1) and (2). That is true for $n \leqslant 2$ in view of Lemma, because every trivial operation satisfies (1) and (2). By induction, let us suppose (1) and (2) for any $g \in A^{(n)}(\mathfrak{A})$, where $n \geqslant 2$, and let $f \in A^{(n+1)}(\mathfrak{A})$. If $(x_1, ..., x_n, x_{n+1})$ is a sequence of 0's and 1's then there are two indices $i \neq j$ such that $x_i = x_j$. Without any loss of generality we may suppose $i = n$ and $j = n+1$. Let us define an operation g of n variables putting for any sequence $y_1, ..., y_n$ of 0's and 1's:

$$g(y_1, ..., y_n) = f(y_1, ..., y_n, y_n).$$

We have $g \epsilon A^{(n)}(\mathfrak{A})$ (see [1], p. 732, 1 (iii)), whence, applying (1) to operation g,

$$f(x_1', \ldots, x_n', x_{n+1}') = f(x_1', \ldots, x_n', x_n') = g(x_1', \ldots, x_n') = [g(x_1, \ldots, x_i)]' =$$
$$= [f(x_1, \ldots, x_u, x_n)]' = [f(x_1, \ldots, x_n, x_{n+1})]'.$$

The passage from n to $n+1$ for the condition (2) is easy.

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MATHÉMATIQUE

Sur la dérivabilité de la limite d'une suite de fonctions possédant une dérivée approximative unilatérale (cas de l'espace de Banach)

par

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Présenté le 17 mars, 1960

Les théorèmes qui suivent constituent une généralisation d'un théorème que nous avons antérieurement communiqué sans en donner une démonstration (cf. [3]). Au cas des fonctions $f_n(x)$ (cf. les notations qui suivent) réelles et dérivables, au sens ordinaire, dans l'intervalle J, les théorèmes III et IV de l'ouvrage precité [3] coïncident avec un théorème classique. Au cas que nous traitons dans la suite, il y a lieu s'appuyer sur un théorème des accroissements finis convenablement modifié (cf. p.e. [4] et Lemme 4). Afin de mieux caractériser la généralité de la méthode servant de base aux démonstrations, nous avons énoncé les Théorèmes 1 et 2 en y introduisant les drivées ε-approximatives (notion liée de près à celle du contingent de G. Bouligand). Dans ce cas il peut arriver que les dérivées à droite des $f_n(x)$ n'existent nulle part et la fonction limite possède partout la dérivée continue (Exemple 1). Nos théorèmes fournissent aussi des conditions suffisantes pour la dérivabilité (au sens ordinaire) de la fonction limite, ce qui peut être essentiel pour certaines applications (cf. [5]).

§ 1. Hypothèse préliminaire. Nous désignons par

$$J = (a, a+b \qquad (0 < b < +\infty) \tag{1.1}$$

un intervalle réel, ouvert, borné; par R — un sousensemble dénombrable de J et par B — l'espace linéaire complet de Banach à norme homogène.

§ 2. DEFINITION 1. Soit $q(x)$ une fonction définie dans J. Nous désignerons par $D_+ q(x)$ et $D_{(-)} q(x)$ respectivement les dérivées à droite et à gauche. Au cas de la fonction $q(x)$ aux valeurs réelles nous désignons par $\bar{D}_+ q(x)$, $\bar{D}_{(-)} q$, $\underline{D}_+ q$ et $\underline{D}_{(-)} q$, respectivement, les dérivées supérieures à droite et à gauche et les dérivées inférieures analogues.

Soit $F(x)$ une fonction aux valeurs dans B définie dans J. Soit $d \in B$, $r \in J$, $0 \leqslant k < +\infty$ et soit h une variable réelle positive. Si l'on a

$$\limsup_{h\to 0} |[(F(r+h) - F(r))/h] - d| \leqslant k, \tag{2.1}$$

on dira que d constitue une dérivée k-approximative à droite de F au point $x=r$. Pour h négatif et tendant vers zéro, le d intervenant dans (2.1) constitue une dérivée k-approximative à gauche. Le nombre k peut être appelé la „tolérance" de ces dérivées approximatives.

R e m a r q u e 1. La dérivée k-approximative d (cf. (2.1)) n'est pas toujours définie d'une façon univoque. Au cas $k=0$ l'élement d est égal à la dérivée à droite de F au point $x=r$.

§ 3. On doit à A. Zygmund le lemme suivant: si dans J la fonction réelle $m(x)$ est continue et dans J-R on a $\overline{D}_+ m(x) \geqslant 0$ (ou $\underline{D}_+ m(x) \leqslant 0$), alors $m(x)$ est croissante au sens large (ou décroissante au sens large) dans J (cf. [2], p. 173). On peut en déduire des théorèmes sur les accroissements finis de caractère bien général (cf. [3], [4]) dont les lemmes, qui suivent, constituent une conséquence. En introduisant les fonctions auxiliares $m(x)=L(x)+k(x)$ et $m(x)=L(x)-k(x)$, on déduit directement du lemme de A. Zygmund les lemmes suivants:

LEMME 1. *Si pour une fonction $L(x)$ réelle et continue et pour un k $(0 \leqslant k < +\infty)$ on a dans J-R l'inégalité $|D_+ L(x)| k$, alors on a $|L(x)-$ $-L(y)| \leqslant k|x-y|$ pour tout $x \epsilon J$ et $y \epsilon J$.*

LEMME 2. *Si pour la fonction $s(x)$ aux valeurs dans B, continue dans J, on a pour la variable positive h (ou négative, respectivement)*

$$\limsup |(s(x+h)-s(x))/h| \leqslant k \quad \textit{dans} \quad J-R, \tag{3.1}$$

alors

$$|s(x)-s(y)| \leqslant k\,|x-y| \quad \textit{pour} \quad x \epsilon J,\ y \epsilon J. \tag{3.2}$$

Posons, en effet, pour un y fixe $L(x)=s(x)-s(y)$. Il résulte de l'inégalité $|L(x+h)-L(x)| \leqslant |s(x+h)-s(x)|$ et de (3.1) que $|D_+ L(x)| \leqslant k$ dans J-R d'où (cf. Lemme 1) $|s(x)-s(y)| = |L(x)-L(y)| \leqslant k|x-y|$.

LEMME 3. *Si la fonction $f(x)$ aux valeurs situées dans B est continue dans J et s'il existe une fonction $G(x)$ continue dans J, telle que*

$$D_+ f(x) = G(x) \quad \textit{dans} \quad J-R,$$

alors f possède partout dans J la dérivée (bilatérale) ordnaire $f'(x)$ et $f'(x)=G(x)$ dans J.

D é m o n s t r a t i o n. Soit, en effet, $p \epsilon J$, soit $\varepsilon > 0$ et choisissons $t > 0$ de façon qu'en posant $J(t)=(p-t, p+t)$, on ait $J(t) \subset J$ et que l'on ait $|G(x)-G(p)| \leqslant \varepsilon$ pour $x \epsilon J(t)$. La fonction auxiliaire $s(x)=f(x)-$ $-xG(p)$ est continue dans $J(t)$ et l'on a $|D_+ s(x)| = |G(x)-G(p)| \leqslant \varepsilon$ dans $J(t)-R$. En vertu du Lemme 2 il s'ensuit que dans $J(t)$ on a $|s(x)-s(p)| \leqslant$ $\leqslant \varepsilon|x-p|$, d'où

$$|(f(x)-f(p))/(x-p)-G(p)| \leqslant \varepsilon \quad \text{pour} \quad x \epsilon J(t)$$

ce qui achève la démonstration.

LEMME 4. (Corollaire du Lemme 2). *Si pour la fonction $s(x)$ aux valeurs dans B, continue dans J, et pour un k $(0 \leqslant k < +\infty)$ on a dans $J-R$ l'inégalité $|D_+ s(x)| \leqslant k$, alors on a $|s(x)-s(y)| \leqslant k|x-y|$ pour tout $x \in J$, $y \in J$.*

§ 4. THÉORÈME 1. *Nous admettons que les valeurs des fonctions $f_n(x)$ et $g_n(x)$ sont situées dans B, que les $f_n(x)$ sont continues dans J et que les $g_n(x)$ sont définies (sans être forcément continues) dans $J-R$ et, qu'enfin*

$$\lim_{h\to 0+0} \sup |(f_n(n+h)-f_n(x))/h - g_n(x)| \leqslant \varepsilon_n(x) < +\infty \quad \textit{dans } J-R, \tag{4.1}$$

$$g(x) \Rightarrow g(x) \quad \textit{dans} \quad J-R, \tag{4.2}$$

$$\varepsilon_n(x) \Rightarrow 0 \quad \textit{dans} \quad J-R, \tag{4.3}$$

$$f_n(c) \to r \in B \quad \textit{pour un certain } c \in J, \tag{4.4}$$

où le symbole $\Rightarrow$ désigne la convergence uniforme.

Ceci posé, nous maintenons qu'il existe une fonction continue dans J, possédant la dérivée à droite dans $J-R$ et telle que

$$f_n(x) \Rightarrow f(x) \text{ dans } J, \tag{4.5}$$

$$D_+ f(x) = g(x) \text{ dans } J-R. \tag{4.6}$$

Si, tout en tenant compte des hypothèses précédentes, on suppose, de plus, qu'il existe une fonction continue dans J tout entier et telle que

$$G(x) = g(x) \text{ dans } J-R, \tag{4.7}$$

alors la dérivée bilatérale $f'(x)$ existe dans J tout entier et l'on a

$$f'(x) = G(x) \text{ dans } J. \tag{4.8}$$

Remarque 1. La relation (4.1) exprime que $g_n(x)$ constitue (dans $J-R$) une dérivée $\varepsilon_n(x)$-approximative à droite de $f_n(x)$ et 4.3) que la suite des tolérances de ces dérivées (cf. Définition 1) tend uniformément vers zéro dans $J-R$.

Démonstration. Soit $\varepsilon > 0$. Choisissons (cf. (4.2), (4.3), (4.4)) l'indice N de façon que l'on ait pour

$$i \geqslant N, \quad j \geqslant N, \quad x \in J-R \tag{4.9}$$

les relations

$$|g_i(x)-g(x)| \leqslant \varepsilon, \quad |g_i(x)-g_j(x)| \leqslant \varepsilon, \ \varepsilon_i(x) \leqslant \varepsilon, \quad |f_i(c)-f_j(c)| \leqslant \varepsilon. \tag{4.10}$$

Dans la suite nous admettons que (4.9) a lieu. Pour une $F(x)$ donnée posons

$$Q(F,x,h) = (F(x+h)-F(x))/h.$$

Soit

$$A_{ij}(x) = f_i(x) - f_j(x).$$

On a alors l'inégalité évidente

$$|Q(A_{ij}, x, h)| \leqslant |Q(f_i, x, h) - g_i(x)| + |Q(f_j, x, h) - g_j(x)| + |g_i(x) - g_j(x)|$$

donc (cf. (4.9), (4.1), (4.10)),

$$\limsup_{h \to 0+0} |Q(A_{ij}, x, h)| \leqslant 3\varepsilon \qquad (\text{pour} \quad x \in J - R)$$

et par consequent (cf. Lemme 4) pour $x \in J$, $y \in J$

$$|A_{ij}(y) - A_{ij}(x) \leqslant |y - x| 3\varepsilon \leqslant 3b\varepsilon,$$

où b est la longueur de J. On a donc

$$(4.11) \qquad |f_i(y) - f_j(y) - f_i(x) + f_j(x)| \leqslant |y - x| 3\varepsilon \leqslant 3b\varepsilon.$$

En y posant $y = c$, on obtient dans J (cf. (4.10))

$$|f_i(x) - f_j(x)| \leqslant 3b\varepsilon + |f_i(c) - f_j(c)| \leqslant (3b+1)\varepsilon.$$

Il en résulte que la convergence uniforme (4.5) a lieu et que $f(x)$ est continue dans J (vu que les $f_i(x)$ le sont). Le passage à la limite $i \to \infty$ appliqué à (4.11) donne (pour $x, y \in J$, $j \geqslant N$)

$$|f(y) - f(x) - f_j(y) + f_j(x)| \leqslant |y - x| 3\varepsilon.$$

En y posant $y = x + h$, on obtient pour $x \in J - R$, $i \geqslant N$

$$|Q(f, x, h) - g(x)| \leqslant |Q(f_i, x, h) - g_i(x)| + |g_i(x) - g(x)| + 3\varepsilon,$$

Nous avons en consequence (cf. (4.1), (4.10)) pour $x \in J - R$

$$\limsup_{h \to 0+0} |Q(f, x, h) - g(x)| \leqslant 5\varepsilon,$$

ce qui entraîne la relation (4.6). Pour démontrer (4.8) il suffit d'appliquer le Lemme 3.

Théorème 2. *Le Théorème 1 reste vrai lorsque l'on y remplace la limité supérieure à droite* (4.1) *et la dérivée à droite* (4.6) *par la limite supérieure et la dérivée à gauche.*

Théorème 3. *Remplaçons dans le Théorème 1 les prémisses* (4.1), (4.2), (4.3) *par l'hypotèse que $D_+ f_n(x)$ existe dans J—R et que*

$$D_+ f_n(x) \Rightarrow g(x) \quad \textit{dans} \quad J - R.$$

La thése du Théorème 1 reste vraie.

Théorème 4. *Un théorème analogue au précédent est vrai pour les derivées à gauche $D_{(-)} f_n(x)$.*

EXEMPLE 1. Alexiewicz a construit une fonction $H(x)$ aux valeurs situées dans un espace B qui vérifie dans l'intrevalle J la condition de Lipschitz $|H(x)-H(y)| \leqslant |x-y|$ et qui n'admet pas de dérivée unilatérale pour aucun $x \epsilon J$.

Soit $p \epsilon B$ un point fixe et posons $a_n = 1/n$, $f_n(x) = x^2 p + a_n H(x)$.

La dérivée $D_+ f_n(x)$ n'existe évidemment pour aucun $x \epsilon J$. Néanmoins, en posant $g_n(x) = 2xp$, $\varepsilon_n(x) = a_n$ et en choisissant arbitrairement un $c \epsilon J$, on verifie que les prémisses du Théorème 1 se trouvent vérifiées, lorsque l'on pose $g(x) = 2px$. On aura $f(x) = x^2 p$.

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MATHÉMATIQUE

Sur l'existence et l'unicité des intégrales des équations différentielles ordinaires au cas de l'espace de Banach

par

T. WAŻEWSKI

Présenté le 17 mars, 1960

Dans le cas de l'équation (1.1) envisagée dans l'espace de Banach la continuité de $f(x, y)$ ne suffit pas pour l'existence des intégrales ([1], p. 25). Nous montrons que l'inégalité (2.1) introduite par E. Kamke comme condition d'unicité ([3], p. 99) est aussi suffisante pour l'existence des intégrales. Nous approchons l'intégrale cherchée par une suite des lignes brisées composées d'un nombre fini de morceaux et construites par la méthode de Carathéodory, ce qui permet d'éviter l'application de l'axiome du choix. Il est à remarquer que nous ne posons pas que la fonction $h(x, y)$ soit croissante en u (dans le cas de la fonction $h(x, u)$ croissante en u la méthode des approximations successives conduit plus rapidement au but, cf. [7]). Nous allons traiter les problèmes local et global de l'existence des intégrales.

§ 1. Notations. Hypothèses préliminaires. R désigne la classe des nombres réels, B — un espace linéaire complet de Banach à norme homogène, $|z$ — la norme de z, lorsque $z \epsilon B$, et la valeur absolue de z, lorsque $z \epsilon R$.

Nous envisageons les équations différentielles ordinaires

(1.1) $$y' = f(x, y),$$

(1.2) $$u' = h(x, u).$$

Nous admettons que

(1.3) $$x \epsilon R, \; u \epsilon R, \quad h \epsilon R, \; y \epsilon B, \quad f \epsilon B$$

en entendant par cela que les variables x et u, ainsi que les valeurs que prend $h(x, u)$, sont rélles, que y varie dans B et que les valeurs de $f(x, y)$ sont situées dans B.

§ 2. Théorème 1. Prémisses. $f(x, y)$ est continue et bornée dans le cylindre borné

$$V: 0 \leqslant x \leqslant r, \; |y| \leqslant r \quad (0 < r < +\infty);$$

M désigne la borne supérieure de $|f(x,y)|$ dans V. La fonction $h(x,u)$ est continue dans la demi-bande

$$E: 0 \leqslant x \leqslant r, \quad 0 \leqslant u < +\infty.$$

On a $h(x,u) \geqslant 0$ dans $E, h(x,0) = 0$ dans l'intervalle fermé $[0,r]$ et par l'origine $(0,0)$ passe une intégrale unique (identiquement nulle) de l'équation (1.2). Pour $(x,y) \epsilon V$, $(x,z) \epsilon V$ on a

$$(2.1) \qquad |f(x,y) - f(x,z)| \leqslant h(x, |y-z|).$$

Nous posons

$$(2.2) \qquad a = r/M+2) < r$$

et introduisons le cylindre $W \subset V$

$$W: 0 \leqslant x \leqslant a, |y| \leqslant r.$$

THÉSE. *L'Eq.* (1.1) *admet une intégrale (du reste unique) passant par le point* $x=0$, $y=0$. *Cette intégrale est définie dans l'intervalle ouvert à droite* $[0,a)$ *et son diagramme est situé dans* W.

Démonstration I. Nous affirmons que, pour établir l'existence de l'intégrale en question, il suffit de prouver qu'il existe une suite des fonctions $q_n(x)$ jouissant des propriétés suivantes:

PROPRIÉTÉ P_1. On a $q_n(0) = 0$. La fonction $q_n(x)$ est continue dans $[0,a)$ et y possede la dérivée à droite $D_+ q_n(x)$. Le diagramme de $y = q_n(x)$ est situé dans W.

PROPRIÉTÉ P_2. Pour tout $\varepsilon > 0$ il existe un indice N, tel que l'on a dans $[0,a)$ les inégalités

$$(2.3) \qquad |D_+ q_n(x) - f(x, q_n(x))| \leqslant \varepsilon \quad \text{pour} \quad n \geqslant N,$$

$$(2.4) \qquad |D_+ q_i(x) - D_+(q_j(x))| \leqslant \varepsilon \quad \text{pour} \quad i \geqslant N, j \geqslant N.$$

En effet, il résulte de (2.4) que l'on a dans $[0,a]$ la convergence uniforme vers une certaine fonction $p(x)$

$$(2.5) \qquad D_+ q_n(x) \Rightarrow p(x) \quad \text{dans} \quad [0,a).$$

Vu que $q_n(0) = 0$ et les $q_n(x)$ sont continues, il existe (cf. [6]) une fonction continue $q(x)$, telle que

$$(2.6) \qquad q(0) = 0 \quad q(x) \Rightarrow q(x) \quad \text{dans} \quad [0,a), \; D_+ q(x) = p(x) \quad \text{dans} \quad [0,a)$$

Le cylindre W étant fermé, il s'ensuit de P_1 et de (2.6) que le diagramme de $y = q(x)$ est situé dans W. En vertu de (2.2) et (2.5) on a

$$(2.7) \qquad (x, q_n(x)) \Rightarrow p(x) \quad \text{dans} \quad [0,a),$$

Mais en vertu de la continuité de $f(x,y)$ on obtient de (2.6) la convergence ordinaire $f(x, q_n(x)) \to f(x, q(x))$, donc, en raison de (2.7) et (2.6),

$$(2.8) \qquad D_+ q(x) = f(x, q(x)) \quad \text{dans} \quad [0,a).$$

Il s'ensuit que $D_+q(x)$ est continue dans $[0, a)$, donc (cf. [6]) $q(x)$ admet dans cet intervalle la dérivée bilatérale $q(x)$ identique à $D_+q(x)$ et, par suite (cf. (2.8)), $q'(x) = f(x, q(x))$ dans $[0, a)$ c'est-à-dire $q(x)$ est une intégrale de (1.1) jouissant des propriétés en question.

Il reste donc à établir l'existence d'une suite des fonctions $q_n(x)$ possédant les propriétés P_1 et P_2.

II. Désignons par $t(s)$ le maximum de la fonction continue et non négative $h(x, u)$ dans le rectangle $0 \leqslant s \leqslant a$, $0 \leqslant u \leqslant s$. La fonction $t(s)$ est évidemment continue et croissante dans l'intervalle $0 \leqslant s < +\infty$ et on a $t(0) = 0$, $h(x, u) \leqslant t(u)$ pour $(x, u) \epsilon E$.

III. Considérons le cône $C: 0 \leqslant x \leqslant a$, $|y| \leqslant Mx$. On a $C \subset W$, car — pour $(x, y) \epsilon C$ — il y a $|y| \leqslant Ma < r$.

IV. Posons $c_n = a/n$ $(n = 1, 2, \ldots)$. L'intervalle $[0, a]$ est la somme des n intervalles contigus fermés

$$J(k,n) = [kc_n, (k+1)c_n], \quad (k = 0, 1, \ldots, n-1). \tag{2.9}$$

Fixons l'attention sur un $J(k, n)$. Soit $\bar{y} \epsilon B$ un point quelconque, tel que le point $x = k c_n$, $y = y$ appartient au cône C. On a $|y| \leqslant k Mc_n$. En appliquant une idée due à C. Carathéodory ([2], pp. 13—15), posons

$$q_{k,n}(x, \bar{y}) = \bar{y} + \int_{kc_n}^{x} f(z, \bar{y})\, dz \quad \text{pour} \quad x \epsilon J(k, n).$$

Le diagramme de $y = q_{k,n}(x, \bar{y})$ est situé dans C (et dans W, cf. III) car dans $J(k, n)$ on a $q_{k,n}(x, y) \leqslant Mkc_n + (x - kc_n)M = Mx$. Introduisons l'intervalle ouvert à droite $L(k, n) = [kc_n, (k+1)c_n)$; on a

$$|q_{k,n}(x, \bar{y}) - \bar{y}| \leqslant M(x - kc_n) \leqslant Mc_n \quad \text{dans} \quad J(k, n) \tag{2.10}$$

et par suite (cf. II) on a dans $L(k,n)$

$$|D_+ q_{k,n}(x, \bar{y}) - f(x, q_{k,n}(x, \bar{y}))| = |f(x, \bar{y}) - f(x, q_{k,n}(x, \bar{y}))| \leqslant$$
$$\leqslant h(x, |\bar{y} - q_{k,n}(x, y)|) \leqslant t(|\bar{y} - q_{k,n}(x, y)|)$$

d'où (cf. II et (2.10))

$$D_+ q_{k,n}(x, \bar{y}) - f(x, q_{k,n}(x, \bar{y})| \leqslant t(Mc_n).$$

Observons enfin que $q_{k,n}(x, y)$ est continue dans $J(k, n)$ et que $q_{k,n}(c_n, \bar{y}) = \bar{y}$.

V. Posons maintenant $q_n(x) = q_{0,n}(x, 0)$ dans $J(0, n)$. En supposant que $q_n(x)$ est définie dans $J(k, n)$ nous posons $y_{k+1} = q_n((k+1)c_n)$ et $q_n(x) = q_{k+1,n}(x, y_{k+1})$ dans $J(k+1, n)$. En vertu de IV $q_n(x)$ est continue dans $[0, a]$, $q_n(0) = 0$,

$$|D_+ q_n(x) - f(x, q_n(x)| \leqslant t(Mc) \quad \text{dans} \quad [0, a). \tag{2.11}$$

VI. Désignons par $s_n(x)$ l'intégrale supérieure issue de point $x = 0$, $u = 0$ de l'équation

$$u' = h(x, u) + 2t(Mc_n).$$

Le deuxième membre de cette équation est non négatif dans la demi-bande E et $2t(Mc_n) \to 0$. Mais l'Eq. (1,2) admet $u(x) = 0$ comme l'unique intégrale issue du point $x = 0$, $u = 0$. Il s'ensuit (cf. [3], p. 83) que les integrales $s_n(x)$ existent dans $[0, a]$ à partir d'un certain indice et l'on a $s_n(x) \Rightarrow 0$ dans $[0, a]$.

Soit $\varepsilon > 0$. Pour un certain N on a

$$(2.12) \qquad 0 \leqslant s_n(x) \leqslant \varepsilon \quad \text{dans} \quad [0, a] \text{ pour } n \geqslant N.$$

VII. Soit $i \geqslant N$, $j \geqslant N$. On a dans $[0, a)$ l'inégalité facile à établir (cf. [5], p. 3) $D_+|q_i - q_j| \leqslant |D_+q_i - D_+q_j|$, où le premier membre désigne la dérivée supérieure de la fonction réelle $|q_i - q_j|$. Mais

$$|D_+q_i - D_+q_j| \leqslant |D_+q_i - f(x, q_j)| + |D_+q_j - f(x, q_i)| + |f(x, q_i) - f(x, q_j)|.$$

En vertu de (2.11), des inégalités $c_i \leqslant c_N$, $c_j \leqslant c_N$ et de la croissance de $t(s)$ on obtient

$$\overline{D}_+|q_i - q_j| \leqslant t(Mc_i) + t(Mc_j) + h(x, q_i - q_j) \quad \text{et} \quad D_+|q_i(x) - q_j(x)| \leqslant$$
$$\leqslant 2\,t(Mc_N) + h(x, |q_i(x) - q_j(x)|) \text{ dans } [0, a).$$

Comme $|q_i(x) - q_j(x)|$ est continue et comme $|q_i(0) - q_j(0)| = 0$, il s'ensuit en vertu de (2.12) (cf. [4], p. 124) que $|q_i(x) - q_j(x)| \leqslant s_N(x)$ dans $[0, a)$. L'inégalité (2.4) a donc lieu. L'inégalité (2.3) résulte de (2.11). Ainsi nous avons prouvé que, pour la suite des fonctions $q_n(x)$, les propriétés P_1 et P_2 ont lieu. Il existe donc une intégrale de (1.1) issue du point $x = 0$, $y = 0$ et définie dans l'intervalle $[0, a)$. Pour démontrer l'unicité envisageons deux intégrales $y = q(x)$ et $y = r(x)$. Un raisonnement analogue a celui qui à été appliqué plus haut montre que $D_+|q - r| \leqslant$ $\leqslant |f(x, q) - f(x, r)| \leqslant h(x, |q - r|)$ dans $[0, a)$. Etant donné que $|q(0) - r(0)| = 0$ et $|q(x) - r(x)|$ est continu dans $[0, a)$, on a, dans cet intervalle, $|q - r| \leqslant 0$, car la fonction identiquement nulle constitue l'intégrale supérieure issue de $x = 0$, $u = 0$ de l'Eq. (1.2). On a donc $q(x) = r(x)$ dans $[0, a)$.

On peut facilement déduire du Théorème 1 le Théorème suivant

THÉORÈME 2 (de caractère local). *Conservons l'hypothèse préliminaire du § 1. Admettons que $f(x, y)$ soit continue dans un petit voisinage $|x - x^*| \leqslant c$, $|y - y^*| \leqslant c$ du point $P = (x^*, y^*)$, que $h(x, u)$ soit continue, non négative dans un petit voisinage $0 \leqslant x \leqslant m$, $0 \leqslant u \leqslant m$ du point $x = 0$, $u = 0$ et que l'Eq. (1.2) admette, comme l'unique intégrale issue de l'origine, la solution identiquement nulle. Admettons enfin que $|f(x, z) - f(x, y)| \leqslant h(|x - x^*|, |z - y|)$ dans lu petit vosinage de P. Ceci posé, on voit que l'Eq.* (1.1) *admet une intégrale (localement unique) issue de P et définie dans un petit voisinage bilatéral de x^*.*

THÉORÈME 3. P r é m i s s e s. $0 < R \leqslant +\infty$, $0 < T \leqslant +\infty$, $h(x, u)$ est continue et non négative dans l'ensemble $E(R, T): 0 \leqslant x < R$, $0 \leqslant u < T$. On a $h(x, 0) = 0$ dans $[0, R)$ et l'Eq. (1.2) admet l'intégrale identiquement

nulle, comme l'unique intégrale issue de l'origine. $f(x, y)$ est continue dans un ensemble ouvert englobant l'ensemble $V(R, T) : 0 \leqslant x < R,\ |y| < T$. L'équation $u' = f(x, 0)| + h(x, u)$ admet $u = U(x)$ comme intégrale supérieure issue de l'origine. $U(x)$ existe dans $[0, c)$, $(0 \leqslant c \leqslant R)$ et on a $U(x)$ T dans $[0, c)$.

THÈSE. *L'Eq.* (1.1) *admet une intégrale unique* $y = q(x)$ *issue de* $x = 0$, $y = 0$. *Cette intégrale se laisse prolonger sur* $[0, c)$ *et* $|q(x)| \leqslant U(x)$ *dans* $[0, c)$.

Démonstration. Soit $[0, k)$, $(k \leqslant c)$ l'intervalle maximum dans lequel l'intégrale $y = q(x)$, $(q(0) = 0)$ de (1.1) existe. On a (cf. Théorème 2) $0 < k$, et $D_+ |q(x)| \leqslant |q'(x)| \leqslant |f(x\ 0)| + h(x, |u(x)|)$ et par suite $|q(x)| \leqslant \leqslant U(x)$ dans $[0, k)$. Il reste à prouver que $k = c$. Supposons que $k < c$. Soit N le maximum de $|f(x, 0)| + h(x, u)$ dans $Z : 0 \leqslant x \leqslant k, 0 \leqslant u \leqslant U(x)$. On a $|q'(x)| \leqslant N < +\infty$ dans $[0, k)$ donc $\lim\limits_{x \to k-0} (x) = m$ existe et en posant $q(k) = m$ on prouve que la derivée à gauche $q'(k)$ existe et que $q'(k) = = f(k, q(k))$. Soit K le point $x = k$, $y = q(k)$. On a $K \in V(R, T)$. Par ce point (cf. Théor. 2) passe une intégrale définie dans $[k, k+\varepsilon)$, donc l'intégrale $y = q(x)$ de (1.1) peut être prolongée sur l'intervalle $[0, k+\varepsilon)$, ce qui est incompatible avec la définition de k.

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MATHEMATICS

On the Congruence $a^x \equiv b \pmod p$

by

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Presented by W. SIERPIŃSKI on March 17, 1960

The aim of this paper is to prove the following theorem signalled in [1].

THEOREM. *If a, b are rational integers, $a > 0$ and $b \neq a^k$ (k — rational integer), then there exists an infinite number of rational primes, p, for which the congruence $a^x \equiv b \pmod p$ has no solutions in rational integers x.*

LEMMA. *Let l be an arbitrary rational prime, ζ_l — a primitive root of unity of l degree, $k = \Gamma(\zeta_l)$ — the field obtained by adjoing ζ_l to the field Γ of rational numbers. If a system of rational integers $\gamma_1, \gamma_2, \ldots, \gamma_t$ has the property, that $\gamma^{m_1}, \gamma^{m_2}, \ldots, \gamma^{m_t}$ is a l-th power of a rational integer only when $l | m_i$ $(i = 1,2 \ldots, t)$, then for arbitrary rational integers $c_1, c_2 \ldots, c_t$ there exists an infinite number of prime ideals $\mathfrak{p}$ of the field k, whose degree is 1, and for which*

$$\left(\frac{\gamma}{\mathfrak{p}}\right) = \zeta^{li} \quad (i = 1, 2, \ldots, t)$$

P r o o f o f t h e l e m m a. If number $\gamma_1^{m_1}, \gamma_2^{m_2}, \ldots, \gamma_t^{m_t}$ is not an l-th power of a rational integer, then the polynomial $x^l \gamma_1^{m_1}, \gamma_2^{m_2}, \ldots, \gamma_t^{m_t}$ is irreductible in Γ. On the basis of a well-known theorem ([2] p. 298, th. 16) the polynomial remains irreductible in the field $\Gamma(\zeta_l)$, therefore $\gamma_1^{m_1}, \gamma_3^{m_2}, \ldots, \gamma_t^{m_t}$ is not an l-th power of the integer of the field k. The thesis of the lemma follows from this directly, in view of Čebotarew's improvement of a Hilbert's theorem ([3] cf. [4] p. 276, th. 152).

P r o o f o f t h e T h e o r e m. The cases $a = 1$ and $b = 0$ are trivial. Assume that $a > 1$, $b \neq 0$, hence $|ab| > 1$ and let $q_1, q_2, \ldots, q_s$ be all the prime factors of ab.

Let further

$$a = q_1^{\alpha_1}, q_3^{\alpha_2}, \ldots, q_s^{\alpha_s}, \qquad b = \pm q_1^{\beta_1}, q_2^{\beta_2}, q_s^{\beta_s}, \qquad (\alpha_i, \beta_i \geqslant 0).$$

If $b < 0$, we observe that the numbers $l = 2$, $t = s+1$, $\gamma_i = q_1$ $(i = 1, 2, \ldots, s)$, $\gamma_{s+1} = -1$ satisfy the conditions of our Lemma. Then,

there exists an infinite number of prime ideals $\mathfrak{p}$ of the field $\Gamma(-1)$ (i.e. simply rational primes), for which we have

$$\left(\frac{q_i}{\mathfrak{p}}\right)=1, \left(\frac{-1}{\mathfrak{p}}\right)=-1,$$

whence

$$\left(\frac{a}{\mathfrak{p}}\right)=1, \left(\frac{b}{\mathfrak{p}}\right)=-1.$$

Assume now, that $b>0$ and that for some indices $i,j\leqslant s$ we have $\alpha_i\beta_j-\beta_i\alpha_j\neq 0$. Choose a rational prime $l>|\alpha_i\beta_j-\beta_i\alpha_j)$.

The numbers $\gamma_i=q_i\,(i=1,2,\ldots s)$, as different rational primes and the number l satisfy the conditions of the Lemma. Then there exists an infinite number of prime ideals $\mathfrak{p}$ of the field $\Gamma(\zeta_l)$, the degree of which is 1 and for which we have

$$\left(\frac{q_\nu}{\mathfrak{p}}\right)=1\ (\nu\neq i,j), \quad \left(\frac{q_i}{\mathfrak{p}}\right)=\zeta^{-\alpha_j}, \quad \left(\frac{q_j}{\mathfrak{p}}\right)=\zeta^{\alpha_i},$$

whence

$$\left(\frac{a}{\mathfrak{p}}\right)-1, \left(\frac{b}{\mathfrak{p}}\right)=\zeta^{\alpha_i\beta_j-\beta_i\alpha_j}\neq 1.$$

In both considered cases, therefore, there exists such a rational prime l, that the field $\Gamma(\zeta_l)$ contains infinitely many prime ideals $\mathfrak{p}$, of the degree 1, for which $\left(\frac{b}{\mathfrak{p}}\right)\neq 1$, but $\left(\frac{a}{\mathfrak{p}}\right)=1$, whence $\left(\frac{a^x}{\mathfrak{p}}\right)=1$, then the congruence $a^x=b\pmod{\mathfrak{p}}$ is insoluble.

The same property has, of course, the congruence $a^x\equiv b\pmod{p}$, where the rational prime p is the norm of ideal $\mathfrak{p}$. As a prime p can be a norm of only a finite $\leqslant 1$ number of prime ideals, there exists in the considered cases an infinite number of rational primes, for which the congruence $a^x\equiv b\pmod{p}$ is insoluble.

We have still to examine the case, when $b>0$ and when for all $i,j\leqslant s$: $\alpha_i\beta_j-\alpha_j\beta_i=0$. As $\alpha_i+\beta_i>0$ $(i=1,2,\ldots,s)$ and not all are $=0$, it follows from the last formula, that all α_i are $\neq 0$ and that for $i\leqslant s$ $\frac{\beta_i}{\alpha_i}=\frac{\beta_i}{\alpha_1}$ holds.

Let

$$\frac{\alpha_1}{(\alpha_1,\beta_1)}=\alpha, \quad \frac{\beta_1}{\alpha_1,\beta_1)}=\beta.$$

As $(\alpha,\beta)=1$, $\frac{\beta_i}{\alpha_i}\frac{\beta}{\alpha}$ $(i\leqslant s)$, we have $\alpha_i=\alpha\delta_i$, $\beta=\beta\delta_i$, where δ_i-are positive integers.

Putting $c=q_1^{\delta_1}\ q_2^{\delta_2},\ldots,q_s^{\delta_s}$, we get $a=c^\alpha$, $b=c^\beta$.

If $\alpha=1$, one obtains $b=a^\beta$, in spite of the conditions assumed. Hence, $\alpha>1$ and there exists a rational prime $l|\alpha$. Choose a positive integer h

so that $l^h | 2(\delta_1, \delta_2, \ldots, \delta_s)$. As the numbers q_i are primes, it follows from the last formula that c is neither of the form n^{l^n} nor of the form $2^{l^{h,2}} n^{l^h}$, where n is the rational integer. By the Trost's theorem ([5]) there exists, therefore, an infinite set P of rational primes p, for which c is not a residue of l^h — th degree.

As $l \nmid \alpha$, $l \mid \beta$ for any rational integer x: $l \nmid \alpha x - \beta$. Hence, for all $p \varepsilon P$, for all x: $c^{\beta x - \beta}$ is not a residue of l^h-th degree, then, the congruence $c^{\alpha x} \equiv c^{\beta}$ (mod p) i.e. the congruence $a^x \equiv b$ (mod p) is impossible.

This completes the proof.

Remark. The Theorem remains true, if $a > 0$ is not assumed, but the proof is longer.

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MATHEMATICS

On the Equation $2^n - D = y^2$

by

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We have proved in [1] that the diophantine equation $2^x - 1 = \frac{1}{2} y \times (y+1)$ has only four solutions in positive integers $(x, y) = (1, 1), (2, 2), (4, 5)$ and $(12,90)$. This equation is equivalent to the $2^{x+3} - 7 = (2y+1)^2$, where $x \geqslant 1$ and $y \geqslant 1$. Hence, the diophantine equation $2^n - 7 = z^2$ has only five solutions in the positive integers $(n, z) = (3,1), (4,3), (5,5), (7,11), (15,181)$. The same result has been obtained by Th. Skolem, S. Chowla and D. J. Lewis in paper [2] by other methods. In the present paper we apply our method to obtain the solutions of the diophantine equation

$$2^n - D = y^2 \tag{1}$$

in positive integers n, y. Our method enables us to find all solutions of (1), if 1) $D \not\equiv 0, 4, 7 \bmod 8$, or 2) D is Mersenne number, i.e. $D = 2^a - 1$ $(a = 1, 2, \ldots)$, or 3) there exists such a prime number p, that $p \mid D$ and $p \equiv \pm 3 \bmod 8$. All solutions of (1) if D satisfies 1) are also given in [2], but our proof is quite elementary. We have proved that if $D \equiv 7 \bmod 8$ and $D > 0$, then the obtaining of all solutions of (1) is equivalent to the investigation of some recurring sequence. At the end of the paper all solutions of (1) are given, if D does not satisfy 1), 2) and 3) and $0 < D \leqslant 150$. We have found no positive number $D \equiv 7 \bmod 8$ (excluding Mersenne numbers and $D = 23$) such that (1) have more than one solution. It may be supposed that such a number does not exist.

THEOREM 1. *If $D \not\equiv 0,4,7 \bmod 8$, then Eq. (1) has one solution at the most. If any solution exists, then $n \leqslant 2$.*

P r o o f. We have $y^2 \equiv 0, 1, 4 \bmod 8$ and $D \equiv 1, 2, 3, 5, 6 \bmod 8$. Hence, $2^n = y^2 + D \not\equiv 0 \bmod 8$ and $n \leqslant 2$. Let $(1, y_1)$ and $(2, y_2)$ be the solutions of (1). Then, $2 - D = y_1^2$ and $4 - D = y_2^2$. Subtracting, we obtain $2 = y_2^2 - y_1^2$, and this is impossible.

If $4 \mid D$, then, in spite of (1), we have $2 \mid y^2$ and $2 \mid y$. Thus, $2^{n-2} - \frac{D}{4} = \left(\frac{y}{2}\right)^2$. We obtained the equation of type (1). If $4 \left| \frac{D}{4} \right.$, this procedure can be con-

tinued (after verifying if $n=2$ gives the solution of (1)), until the equation $2^{n-2k}-\frac{D}{4^k}=\left(\frac{h}{2^k}\right)^2$ is obtained, where $4\left|\frac{D}{4k}\right.$. It follows that we can limit ourselves to the case $D\equiv 7 \bmod 8$.

LEMMA. *Let D be an odd positive integer and $D=d_1 d_2=d_3 d_4$, where $d_1<d_2$, $d_3<d_4$, $d_1<d_3$ and $d_1+d_2=2^a$, $d_3+d_4=2^b$. Then D is the Mersenne number $D=2^a-1$.*

Proof. We have $d_2=\frac{D}{d_1}>\frac{D}{d_3}=d_4$. Hence, $(d_1 d_3)^2<d_1 d_2 d_3 d_4=D^2$ and $d_1 d_3<D$. Then $(d_1+d_2)-(d_3+d_4)=d_1-d_3+\frac{D}{d_1}-\frac{D}{d_3}=(d_3-d_1)\left(\frac{D}{d_1 d_3}-1\right)$ and we infer $a>b$.

From $d_4>d_3>d_1$ we obtain

$$2^{b-1}<d_4<d_4+d_1<2^b. \tag{2}$$

We also have

$$d_1(2^a-d_1)=d_4(2^b-d_4). \tag{3}$$

Hence, $d_4^2-d_1^2\equiv 0 \bmod 2^b$, because $a>b$. Numbers d_1 and d_4 are odd, and $(d_1+d_4,\ d_1-d_4,\ 2^b)=2$. Then one of the numbers d_1+d_4, d_1-d_4 is divisible by 2^{b-1} and the other by 2. In virtue of (2) $2^{b-1}|d_1+d_4$. Hence, $2^{b-1}|d_1+d_4<2^b$ and we infer $d_1+d_4=2^{b-1}$. Since $(d_1, d_4)=1$ we obtain from (3) $d_1\ d_3=2^b-d_4=2^{b-1}+d_1$. Then $d_1\ 2^{b-1}$. Since, however, d_1 is odd, $d_1=1$ and $d_2=2^a-1=D$.

THEOREM 2. *If positive integers n, y satisfy (1) and n is even, then $n\leqslant 2\log_2|D|$. If, moreover, D is an odd positive integer but not a Mersenne number ($D\neq 2^a-1$), then there are no more such solutions of (1) in which n would be even.*

Proof. Let $n=2k$. We have $D=(2^k+y)(2^k-y)$. Hence $2^k\pm y\leqslant|D|$ and $2^k\leqslant|D|$, whence $n=2k\leqslant 2\log_2|D|$. Let D fulfill the conditions of the second part of the theorem and let the Eq. (1) have at least two solutions $n_1=2k_1$, y_1 and $n_2=2k_2$, y_2. We shall then have $D=(2^{n_1}-y_1)\cdot(2^{n_1}+y_1)=(2^{n_2}-y_2)(2^{u_2}+y_2)$ and this is in contradiction with the lemma.

COROLLARY 1. *If such a prime number p exists, that $p\equiv\pm 3 \bmod 8$ and $p|D$ and positive integers n, and y are the solution of (1), then the hypothesis of Theorem 2 is fulfilled.*

Suffices to show that n is even. If $n=2k+1$, then $2^{2k+1}\equiv y^2 \bmod p$ and 2 would be the quadratic residue mod p. This is impossible, because $p\equiv\pm 3 \bmod 8$.

Remark 1. Without the assumption $D>0$ the second part of the Theorem 2 is not true, as shown by the following example $-105=2^4-11^2=2^8-19^2$.

THEOREM 3. *Let D be a squarefree positive integer, $D\equiv 7 \bmod 8$, $\mathfrak{p}=(2, \frac{1}{2}(1+\sqrt{-D}))$ an ideal of the quadratic field $K(\sqrt{-D})$, g — the*

least positive integer such that ideal p^g is principal. Then the positive integers n and y satisfy (1) *if and only if*

(a) $g \mid n-2 > 0$,

(b) $\xi = \sqrt{2^{g+2} - D}$ *is a positive integer*,

(c) $|u_m| = 1$, *where* $m = \dfrac{n-2}{g}$ *and* $u_0 = 0$, $u_1 = 1$,

$$u_{k+2} = \xi u_{k+1} - 2^g u_k \quad \textit{for} \quad k = 0, 1, 2, \ldots .$$

Proof. Let us suppose that the positive integers n and y satisfy (1). It follows from the assumption $D \equiv 7 \bmod 8$ that y is odd and hence $2^n = D + y^2 \equiv 7 + 1 \equiv 0 \bmod 8$. Then $n > 2$. The norm of the ideal p equals 2, because it is easy to see that numbers 0 and 1 are the complete residue system mod p. Hence the ideal p is prime. Let us denote $p' = (2, \frac{1}{2}(1 - \sqrt{-D}))$. We shall then have $(2) = p.p'$. The ideal p' is also prime, because its norm equals 2. Ideals p and p' are co-prime, because $1 = \frac{1}{2}(1 + \sqrt{-D}) + \frac{1}{2}(1 - \sqrt{-D}) \in (p, p')$. Eq. (1) can be written in the following manner: $2^{n-2} = \frac{1}{2}(y + \sqrt{-D})\frac{1}{2}(y - \sqrt{-D})$. Hence, we have the equality for ideals

$$p^{n-2} p'^{n-2} = \left(\frac{1}{2}(y + \sqrt{-D})\right)\left(\frac{1}{2}(y - \sqrt{-D})\right). \tag{4}$$

Ideals $q = (\frac{1}{2}(y + \sqrt{-D}))$ and $q' = (\frac{1}{2}(y - \sqrt{-D}))$ are co-prime, because y and $D \in (q, q')$ and we see from (1) that $(D, y) = 1$, ($D \equiv 7 \bmod 8$ is odd).

From the fundamental theorem of the theory of ideals and from (4) we infer that $p^{n-2} = q$ and $p'^{n-2} = q'$ or $p^{n-2} = q'$ and $p'^{n-2} = q$. Since ideals q and q' are principal, ideals p^{n-2} and p'^{n-2} must be principal too. Hence, $g \mid n-2$. Let us denote $m = (n-2)/g$ and $p^g = (\frac{1}{2}(\xi + \eta\sqrt{-D}))$. We can suppose $\xi \geqslant 0$. We have $N(p^g) = 2^g = \frac{1}{4}(\xi^2 + D\eta^2)$ and hence $\xi = \sqrt{2^{g+2} - D\eta^2}$. Since D is odd, we infer that $\xi > 0$. Now our equalities for ideals can be written in the form:

$$\left\{\begin{array}{l} \left(\frac{1}{2}(\xi + \eta\sqrt{-D})\right)^m = \left(\frac{1}{2}(y + \sqrt{-D})\right) \\ \text{and} \\ \left(\frac{1}{2}(\xi - \eta\sqrt{-D})\right)^m = \left(\frac{1}{2}(y - \sqrt{-D})\right) \\ \text{or} \\ \left(\frac{1}{2}(\xi - \eta\sqrt{-D})\right)^m = \left(\frac{1}{2}(y + \sqrt{-D})\right) \\ \text{and} \\ \left(\frac{1}{2}(\xi + \eta\sqrt{-D})\right)^m = \left(\frac{1}{2}(y - \sqrt{-D})\right). \end{array}\right. \tag{5}$$

As yet we have not employed the fact that D is positive. The only units of the quadratic fields $K(\sqrt{-D})$ discussed here are ± 1. Then from the equalities (5) of principal ideals we obtain the equalities of the generating elements of the ideals (with suitable signs):

$$\pm\left[\frac{1}{2}(\xi+\eta\sqrt{-D})\right]^m=\frac{1}{2}(y+\sqrt{-D})$$

and

$$\pm\left[\frac{1}{2}(\xi-\eta\sqrt{-D})\right]^m=\frac{1}{2}(y-\sqrt{-D})$$

or

$$\pm\left[\frac{1}{2}(\xi-\eta\sqrt{-D})\right]^m=\frac{1}{2}(y+\sqrt{-D})$$

and

$$\pm\left[\frac{1}{2}(\xi-\eta\sqrt{-D})\right]^m=\frac{1}{2}(y-\sqrt{-D}).$$

Substracting the last equalities we shall obtain in any case

$$\sqrt{-D}=\pm\left\{\left[\frac{1}{2}(\xi+\eta\sqrt{-D})\right]^m\pm\left[\frac{1}{2}(\xi-\eta\sqrt{-D})\right]^m\right\}. \tag{6}$$

In the brackets the sign $+$ cannot appear, for then the rational integer would be on the right side of (6). From (6) in the same manner it is easy to deduce that $\eta\neq 0$. Let us denote

$$u_k=\frac{1}{\sqrt{-D}}\left\{\left[\frac{1}{2}(\xi+|\eta|\sqrt{-D})\right]^k-\left[\frac{1}{2}(\xi-|\eta|\sqrt{-D})\right]^k\right\} \quad \text{for } k=0,1,\ldots \tag{7}$$

It is easy to verify that $u_0=0$, $u_1=|\eta|$ and by induction that for $k=0,1,\ldots$, $u_{k+2}=\xi u_{k+1}-\frac{1}{4}(\xi^2+D\eta^2)u_k=\xi u_{k+1}-2^g u_k$. From (6) and (7) we obtain $u_m=\pm 1$. By easy induction it can be shown that $u_1\,|\,u_k$ for $k=0,1,\ldots$. Hence, $|\eta|=u_1\; u_m=\pm 1$, whence $u_1=1$. We have then proved that (a), (b), (c) are satisfied.

Now let us suppose that the conditions (a), (b), (c) are fulfilled. It is easy to prove by induction that every number of the sequence u_k of (c) can be written in the form (7), where $\eta=1$ and ξ is defined by (b). We have also $u_m=\pm 1$. Let us denote $[\frac{1}{2}(\xi+\sqrt{-D})]^m=\frac{1}{2}(w+z\sqrt{-D})$, where w and z are integers. Hence $[\frac{1}{2}(\xi-\sqrt{-D})]^m=\frac{1}{2}(w-z\sqrt{-D})$. Then $\pm 1=u_m=\frac{1}{\sqrt{-D}}[\frac{1}{2}(w+z\sqrt{-D})-\frac{1}{2}(w-z\sqrt{-D})]=z$, and we can write $2^n=4\cdot 2^{gm}=4[\frac{1}{4}(\xi^2+D)]^m=4[\frac{1}{4}(w^2+z^2D)]=w^2+D$.

Hence, positive integers n and $y=|w|$ satisfy (1).

Corollary 2. *If the number g of Theorem 3 is even and $D\neq 2^a-1$, then the Eq. (1) has one solution at the most.*

It follows, indeed, from (a) that $n = gm+2$ is even and number D in Theorem 3 is odd. Then, in virtue of Theorem 2, Eq. (1) has one solution at the most.

Remark 2. If the class number h of the field $K(\sqrt{-D})$ is prime and D satisfies the assumptions of Theorem 3, then $g = h$. Hence, $g > 1$ holds for $D > 7$.

$g|h$ holds for every field $K(\sqrt{-D})$. If $g = 1$, the ideal p would then be principal. The norm of any number $\frac{1}{2}(u + v\sqrt{-D})$ of $K(\sqrt{-D})$ equals $\frac{1}{4}(u^2 + v^2 D)$ and, as was shown in the proof of Theorem 3, $N(p) = 2$. It is easy to see that the equation $\frac{1}{4}(u^2 + v^2 D) = 2$ is solvable in rational integers u, and v only for $D = 7$. Hence, only in this case p could be principal. It is, however, well known that the class number of $K(\sqrt{-7})$ is 1 and it is not a prime.

THEOREM 4. *Let $d = a^2 D$, where a is an odd positive integer, D satisfies the assumptions of Theorem 3, p and g have the same meaning as in the Theorem 3, then the positive integers n and y satisfy the equation $2^n - d = y^2$ if an only if*

(a′) $g | n - 2 > 0$;

(b′) *there exists an integer η such that $\eta | a$ and $\xi = \sqrt{2^{g+2} - D\eta^2}$ is the positive integer;*

(c′) $|u_m| = a$, *where $m = (n-2)/g$, and $u_0 = 0$, $u_1 = \eta$, $u_{k+2} = \xi u_{k+1} - 2^g u_k$ for $k = 0, 1, 2, \ldots$.*

The proof is similar to that of Theorem 3 and will be omitted.

THEOREM 5. *Let D, m, u_k have the same meaning as in Theorem 3. Then $D = 2^{\lambda+2} - 1 > 7$ implies that (1) has only two solutions $(n, y) = (\lambda+2, 1)$, $(2\lambda+2, 2^{\lambda+1} - 1)$. If $D = 2^{\lambda+2}s - 1$, where $s > 1$ is an odd positive integer, and (1) has a solution, then $m = 1$ or $m \equiv s(s - 2^{g-\lambda-1}) \bmod 2^g$.*

Proof. If the sequence v_k is defined by the formulae $v_0 = 0$, $v_1 = 1$, $v_{k+2} = av_{k+1} + bv_k$ for $k = 0, 1, \ldots$, where a, b are rational integers, then (cf. [3])

$$(v_r, v_s) = v_{(r,s)}, \quad \text{whence} \quad v_r | v_{rs} \tag{8}$$

for positive integers r, s.

Moreover, $v_{k+4} = av_{k+3} + bv_{k+2} = a^2 v_{k+2} + ab\, v_{k+1} + b\, v_{k+2} = (a^2 + b)\, v_{k+2} + b(v_{k+2} - b\, v_k) = (a^2 + 2b)\, v_{k+2} - b^2 v_k$ for $k = 0, 1, 2, \ldots$. Then for the sequence u_k

$$u_{k+4} = (\xi^2 - 2^{g+1})\, u_{k+2} - 2^{2g} u_k \quad \text{for} \quad k = 0, 1, \ldots, \tag{9}$$

holds. Formula (9) implies $u_4 = \xi(\xi^2 - 2^{g+1})$. If $|u_4| = 1$, then $|\xi| = 1$ and $|\xi^2 - 2^{g+1}| = |1 - 2^{g+1}| = 1$. This is impossible. Hence, we have $|u_4| \neq 1$. (9) implies for $k = 0, 1, \ldots$, $u_{k+4} \equiv \xi^2 u_{k+2} \equiv u_{k+2} \bmod 4$. Since for $g > 1$

$u_3 = \xi^2 - 2^g \equiv 1 \bmod 4$, we have $u_{2k+1} \equiv 1 \bmod 4$ for $k = 0, 1, \ldots$. The condition $g > 1$ is fulfilled in spite of $D > 7$ and Remark 2. Hence, we infer $u_{2k-1} \neq -1$ and from (8) and $|u_4| \neq 1$ it follows that for $D > 7$ number -1 does not appear in the sequence u_k.

Let us now suppose that $D = 2^{\lambda+2} - 1 > 7$. It is easy to verify that $n = \lambda + 2$ and $y = 1$ satisfy (1). It follows from (a) that $g\ \lambda$ and from (b) that $\sqrt{2^{g+2} - 2^{\lambda+2} + 1} \geqslant 1$. Hence, $\lambda \leqslant g$. It follows that $\lambda = g$ and $\xi = 1$. Formula (9) implies $u_{2k+5} \equiv \xi^2 u_{2k+3} \equiv u_{2k+3} \bmod 2^{g+1}$ for $k = 0, 1, \ldots$. Hence, $u_{2k+3} \equiv u_3 = 1 - 2^g \not\equiv 1 \bmod 2^{g+1}$ for $k = 0, 1, \ldots$, whence $u_{2k-3} \neq 1$. Since every positive integer > 2 has an odd divisor $d > 1$, or is divisible by 4, from (8) and $|u_d| > 1$ we induce that $|u_k| \neq 1$ for $k > 2$. It is easy to verify that $u_1 = u_2 = 1$. This completes the proof of the first part of the Theorem.

Now let us suppose that $D = 2^{\lambda+2} s - 1$, where s is an odd integer > 1, and that (1) has a solution n, y. From condition (b) of Theorem 3 it follows that $\xi^2 = 2^{g+2} - 2^{\lambda+2} s + 1$ and this implies, since $s \geqslant 3$, that $g \geqslant \lambda = 2$ and $\xi > 1$. Hence, the number m of the condition (c) of the Theorem 3 is odd. Let us suppose that there exists such an odd integer $m > 1$ that $u_m = 1$ and let us denote $\frac{1}{2}(m-1) = 2^{\mu} \cdot t$, ($\mu \geqslant 0$, t is odd). The formula (9) implies $1 = u_m \equiv (\xi^2 - 2^{g+1}) u_{m-2} \equiv (\xi^2 - 2^{g+1})^{1/2(m-3)} u_3 \equiv (\xi^2 - 2^{g+1})^{1/2(m-3)} (\xi^2 - 2^g) \equiv (\xi^2 - 2^{g+1})^{1/2(m-1)} + 2^g (\xi^2 - 2^{g+1})^{1/2(m-3)} \bmod 2^{2g}$, therefore

$$(10) \qquad 1 \equiv (1 + 2^{g+1} - 2^{\lambda+2} s)^{1/2(m-1)} + 2^g (1 + 2^{g+1} - 2^{\lambda+2} s)^{1/2(m-3)} \bmod 2^{2g}.$$

It is easy to prove by induction on β the following congruence

$$(11) \qquad (1 + r\, 2^{\alpha})^{2^{\beta} \cdot \gamma} \equiv 1 + r\, 2^{\alpha} \cdot 2^{\beta} \cdot \gamma \bmod 2^{2\alpha+\beta-1},$$

where r, γ are odd positive integers, $\alpha \geqslant 1$, $\beta \geqslant 0$. The formula (11) implies that

$$(12) \qquad (1 + 2^{g+1} - 2^{\lambda+2} s)^{\frac{m-1}{2}} \equiv 1 + 2^{\lambda+2} (2^{g-\lambda-1} - s)^{\frac{m+1}{2}} \bmod 2^{2\lambda+\mu+3}$$

(10) and (12) imply

$$(13) \qquad 1 \equiv 1 - 2^{\lambda+2} s^{\frac{m-1}{2}} + 2^g \bmod 2^{\min(g+1,\, 2\lambda+\mu+3)}.$$

If $g + 1 \leqslant 2\lambda + \mu + 3$, then in spite of (13) it would follow

$$(14) \qquad g = \mu + \lambda + 2.$$

If $g + 1 > 2\lambda + \mu + 3$, then $g > \lambda + \mu + 2$ and from (13) it would follow that $0 \equiv 2^{\lambda+\mu+2} \bmod 2^{\lambda+\mu+3}$, which is impossible. Then it holds (14). From (10), (12), (14) it follows that $1 \equiv 1 + 2^{\lambda+2} (2^{g-\lambda-1} - s) \frac{m-1}{2} + 2^g (1 + 2^{g+1} -$

$-2^{\lambda+2}s - 2^{\lambda+2}s)^{1/2(m-3)} \bmod 2^{g+\lambda+1}$. Therefore, $0 \equiv 2^{\mu+1}t(2^{g-\lambda-1} - s) + 2^{g-\lambda-1}$ mod 2^g and finally $m = 2^{\mu+1}t + 1 \equiv \dfrac{s}{s - 2^{g-\lambda-1}} \bmod 2^g$.

COROLLARY 3. *If* $D = 2^{\lambda+2}s - 1$, $s > 1$ *is odd*, $u_m = 1$ *and* $m > 1$, *then* $m > \frac{1}{2}\sqrt{D}$. *It follows from Theorem* 5 *that* $m \equiv 1 \bmod 2^{g-\lambda-1}$ *and* $m \equiv 1 + 2^{g-\lambda-1} \bmod 2^{g-\lambda}$. *Hence*, $m > 2^{g-\lambda-1}$. *From* (b) *we have* $\xi^2 - 1 = 2^{g+2} - 2^{\lambda+2}s = 2^{\lambda+2}(2^{g-\lambda} - s)$. *Hence, since* $(\xi - 1, \xi + 1) = 2$, *we infer that* $\xi + 1 \geqslant 2^{\lambda+1}$ *and* $\xi - 1 \leqslant 2(2^{g-\lambda} - s)$. *We shall then have* $2^{\lambda+1} \leqslant 2(2^{g-\lambda} - s)$ *and therefore* $s < 2^{g-\lambda}$. *Hence*, $D = 2^{\lambda+2}s - 1 < 2^{g-\lambda}2 \cdot 2(2^{g-\lambda} - s) < (2^{g-\lambda+1})^2 < 4m^2$.

Remark 3. There exist infinitely many such prime numbers D of the form $8t+7$ for which Eq. (1) has no solution. It is easy to verify that the congruence $y^2 + D \equiv 2^n \bmod 255$ has no solution if $D \equiv 14, 26, 56, 104, 131, 134\ 161$, or $224 \bmod 255$. It is also easy to verify that for $D \equiv 5 \bmod 15$, $y^2 + D \equiv 2 \bmod 15$ holds for every positive integer y and n.

We now shall give all solutions of (1) for $0 < D \leqslant 150$. If D satisfies one of the conditions 1), 2) or 3), the method of finding all solutions of (1) is given in Theorems 1 and 2. Then we can investigate the numbers D not satisfying any of the above conditions.

$D = 7$ — all solutions are given in [1] and [2].

$D = 23$, $g = 3$, $\xi = 3$, $\lambda = 1$, $s = 3$. Theorem 5 implies $m = 1$ or $m \equiv 3 \bmod 8$. It is easy to see that $u_3 = 1$. The residues of the sequence u_{8k+3} mod 11 form the periodical sequence with the period 1, 10, 10, 9, 2, 2, 4, 7, 7, 3, 8, 8, 5, 6, 6. Hence, $u_{8k+3} = 1$ implies $k = 15t$. If for any $t > 0$ it would be $u_{120t+3} = 1$, then, since $40t+1 | 120t+3$ and (8), $u_{40t+1} | u_{120t+3} = 1$. Therefore, $u_{40t+1} = 1$ and this is in contradiction with the $m \equiv 3 \bmod 8$.

The only solutions of (1) are $(n, y) = (5, 3)$ and $(11, 45)$.

$D = 47$, $g = 5$, $\xi = 9$, $\lambda = 2$, $s = 3$. Theorem 5 implies $m = 1$ or $m \equiv 29 \bmod 32$ hence $m \equiv 5 \bmod 8$. The residues mod 5 of the terms of the sequence u_{8k+5} form the periodical sequence with the period 4, 4, 2. Hence $u_{8k+5} \neq 1$.

The only solution of (1) $(n, y) = (7, 9)$.

$D = 71$, $g = 7$, $\xi = 21$, $\lambda = 1$, $s = 9$. Theorem 5 gives $m = 1$ or $m \equiv 33 \bmod 128$.

It can be reckoned that $u_{128k+33} \equiv 32 \bmod 127$ and we have the only solution $(9, 21)$.

$D = 79$, $g = 5$, $\xi = 7$, $\lambda = 2$, $s = 5$. Theorem 5 gives $m = 1$ or $m \equiv 5 \bmod 32$. It is easy to see that $u_{8k+5} \equiv 2 \bmod 3$, hence the only solution of (1) is $n = y = 7$.

$D = 103$, $g = 5$, $\xi = 5$, $\lambda = 1$, $s = 13$. Theorem 5 gives $m = 1$ or $m \equiv 9 \bmod 32$.

It follows from $u_{16k+9} \equiv 30 \bmod 31$ that the only solution of (1) is $n = 7$, $y = 5$.

$D = 119$, $g = 5$, $\xi = 3$, $\lambda = 1$, $s = 15$. Theorem 5 implies $m = 1$ or $m \equiv 25 \bmod 32$.

It can be verified that $u_{32k+25} \bmod 97$ is the periodical sequence with the period 31, 58, 56. Therefore, the only solution of (1) is $n = 7$, $y = 3$.

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THEORETICAL PHYSICS

On Optimum Field Homogeneity of High Energy Coil Magnets

by

R. S. INGARDEN and J. MICHALCZYK

Presented by W. RUBINOWICZ on March 8, 1960

In adiabatic demagnetization experiments of low temperature physics as well as in spectroscopy, Hall effect experiments etc. it is very important to have a possibly high magnetic field with a possibly good homogeneity, especially in the axial direction. The first condition may be comparatively easy to satisfy by high energy coil magnets (cf. [1]—[5]), but coils of optimum shape for power consumption (or for the field, if power is given) are notorious for their poor field homogeneity [5]. From the principal point of view the problem of optimum homogeneity with a possibly high field — as far as the present authors know *) — was not discussed in detail in the literature. Only some empirical methods were proposed for improving homogeneity, such as lengthening the coil, leaving a gap in the middle, making steps on the interior side of the coil, inserting thin ferromagnetic plates perpendicular to the axis, etc. For highest fields the second and third only of the named methods are suitable. The first is contrary to the condition of the highest field (cf. also below), the fourth fails above the ferromagnetic saturation field. In the present paper we shall discuss the second method for the simplest case of a cylindrical coil with square ends and uniform current density (case 1 of Bitter's paper [2], confining the discsussion to the field along the axis. On this example, however, we shall formulate general notions and methods of calculation applicable in principle to any case.

An axial section of the coil is presented in Fig. 1 with notations used. We define dimensionless parametres

$$\alpha=\frac{a_2}{a_1}, \quad \beta=\frac{b}{a_1}, \quad \gamma=\frac{c}{a_1}, \quad \xi=\frac{x}{a_1}. \tag{1}$$

*) We do not take the Helmholtz magnet into consideration (cf. e.g. [6], p. 35), which from our point of view is a too ideal case, because of the infinitely small thickness of the wire.

Furthermore, we denote by W the power consumed in the coil, by ϱ — the (constant) specific resistance of the material in which the current is flowing, by λ — the (constant) space factor, i.e. a fraction of the volume of the coil occupied by conducting material, by $H(\xi)$ — the absolute value of the magnetic field along the axis. We consistently use the rationalized Giorgi MKSA system of units (in contradistinction e.g. to [2], where mixed "practical" units are applied).

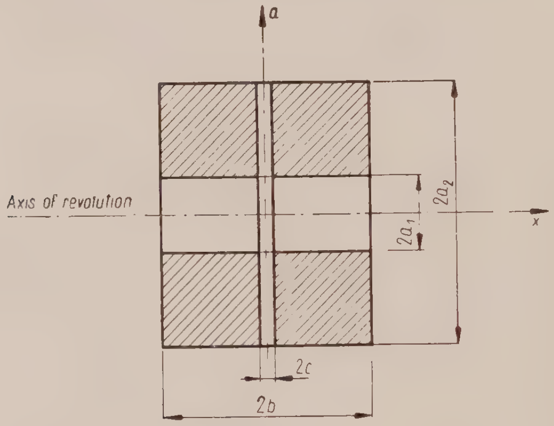

Fig. 1
Locally homogeneous coil: $a_2 = 3.5\,a_1$, $b = 2.8\,a_1$, $c = 0.116\,a_1$

After a direct calculation similar to that in [2] we obtain a generalization of the Fabry formula [7]

$$H(\alpha, \beta, \gamma; \xi) = G(\alpha, \beta, \gamma; \xi) \sqrt{\frac{W\lambda}{\varrho a_1}}, \tag{2}$$

where (for $|\xi| < \beta$)

$$G(\alpha, \beta, \gamma; \xi) = \frac{1}{2\sqrt{2\pi}} \frac{1}{\sqrt{(\beta-\gamma)(\alpha^2-1)}} \left[(\beta+\xi) \ln \frac{\alpha + \sqrt{\alpha^2 + (\beta+\xi)^2}}{1 + \sqrt{1 + (\beta+\xi)^2}} + \right.$$
$$\left. + (\beta - \xi) \ln \frac{\alpha + \sqrt{\alpha^2 + (\beta-\xi)^2}}{1 + \sqrt{1 + (\beta-\xi)^2}} - 2\gamma \ln \frac{\alpha + \sqrt{\alpha^2 + \gamma^2}}{1 + \sqrt{1 + \gamma^2}} \right] \tag{3}$$

is a "geometrical (shape) factor". In the following we consider the "external" parameters W, λ, ϱ, and a_1 as given, and try to determine the "internal" or "shape" parameters from an extremal condition.

As remarked above, the requirements of the highest field and the best homogeneity are contradictory and only a certain compromise between them is possible. Our question is a typical problem of the theory of approximation (cf. e.g. [8]), and it may be formulated in many ways depending on the measure of approximation adopted. What measure will be most suitable in a given case depends on the nature of the experimental problem.

To get these formulations we develop G as function of ξ into the two following expansions:

(i) a Maclaurin series

(4) $$G(\alpha,\beta,\gamma;\xi)=G_0(\alpha,\beta,\gamma)+G_2(\alpha,\beta,\gamma)\,\xi^2+G_4(\alpha,\beta,\gamma)\,\xi^4+\ldots$$

where

(5) $$G_0(\alpha,\beta,\gamma)=G(\alpha,\beta,\gamma;0),\qquad G_{2k}(\alpha,\beta,\gamma)=\frac{1}{(2k)!}\left(\frac{\partial^{2k}G}{\partial\xi^{2k}}\right)_{\xi=0}\qquad k=1,2,\ldots$$

(because of the symmetry of the coil all odd ξ-derivatives of G vanish at $\xi=0$), and

(ii) a Gegenbauer series

$$G(\alpha,\beta,\gamma;\xi)=G_0^{(p)}(\alpha,\beta,\gamma)+G_2^{(p)}(\alpha,\beta,\gamma)\,C_2^{(p)}(\xi)+G_4^{(p)}(\alpha,\beta,\gamma)\,C_4^{(p)}(\xi)+\ldots$$

where $C_{2k}^{(p)}(\xi)$ are the Gegenbauer polynomials (or the Jacobi ultraspherical ones), i.e. polynomials orthogonal in the interval $(-1,+1)$ with the weight function $(1-\xi^2)^{p-\frac{1}{2}}$, $p\geqslant 0$, (cf. [8] Ch. VI) and

$$G_0^{(p)}(\alpha,\beta,\gamma)=A_0\int_{-1}^{+1}\frac{G(\alpha,\beta,\gamma;\xi)}{(1-\xi^2)^{\frac{1}{2}-p}}\,d\xi,\quad G_{2k}^{(p)}(\alpha,\beta,\gamma)=A_{2k}\int_{-1}^{+1}\frac{G(\alpha,\beta,\gamma;\xi)\,C_{2k}^{(p)}(\xi)}{(1-\xi^2)^{\frac{1}{2}-p}}\,d\xi$$

(we assume $\beta\geqslant 1$; A_0, A_{2k} — squares of the normalization constants; here also all odd "Gegenbauer derivatives" of G vanish). In particular, if $p=0$, we have an expansion into the Tchebyshev polynomials of the first kind $T_n(\xi)$ e. g. $T_2(\xi)=2\xi^2-1$, $T_4(\xi)=8\xi^2=1$, $A_0=\pi^{-1}$, $A_2=A_4=2\pi^{-1}$; if $p=\frac{1}{2}$ — into the Legendre polynomials $P_n(\xi)$ (e. g. $P_2(\xi)=\frac{1}{2}(3\xi^2-1$, $P_5(\xi)=$ $=\frac{1}{8}(35\xi^4-20\xi^2+3)$, $A_0=\frac{1}{2}$, $A_2=\frac{5}{2}$, $A_4=\frac{9}{2}$); if $p=1$ — into the Tchebyshev polynomials of the second kind $U_n(\xi)$ (e. g. $U_2(\xi)=4\xi^2-1$, $U_4(\xi)=16\xi^4-$ $-12\xi^2+1$, $A_0=A_2=A_4=2\pi^{-1}$.

Now we define:

(i) the *locally homogeneous magnet* as a coil with such α,β,γ that

(8) $$G_0(\alpha,\beta,\gamma)=\max,$$

by supplementary condition

(9) $$G_2(\alpha,\beta,\gamma)=0;$$

and the *locally superhomogeneous magnet* as determined by (8) with (9) and additionally by

(10) $$G_4(\alpha,\beta,\gamma)=0.$$

Such coils have the best smoothness of the field in the neighbourhood of the point $\xi=0$ (tangency of the fourth or the sixth order, respecti-

vely *), but the absolute deviation of the field from its maximum value at $\xi = 0$ may be large in points $\xi = \pm 1$. These coils are best suited for cases, where objects of investigation are small, but the smoothness of the field must be of the best quality (spectroscopic measurements?). Superhomogeneous magnets have much better homogeneity than homogeneous ones, but at the cost of the field, which is considerably smaller in the former (here the principle of lengthening the coil is applied).

(ii) The *integrally p-homogeneous magnet* for the ξ-interval $(-1, +1)$ **) is determined by

$$G_0^{(p)}(\alpha, \beta, \gamma) = \max, \tag{11}$$

and by the supplementary condition

$$G_2^{(p)}(\alpha, \beta, \gamma) = 0, \tag{12}$$

whereas the *integrally p-superhomogeneous magnet* additionally by

$$G_4^{(p)}(\alpha, \beta, \gamma) = 0. \tag{13}$$

The sense of the approximation is here such, (cf. [8], p. 321), that

$$\int_{-1}^{+1} \frac{[9(\xi) - 9_0^{(p)}]^2}{(1-\xi^2)^{\frac{1}{2}-p}}\, d\xi = \min. \tag{14}$$

among functions given by expansion (6) to the second or fourth order term inclusive, respectively. Except this "least square approximation", solutions for $p = 0$ and $p = 1$ have particular extremal meanings. In the first case we have (cf. [8], p. 63)

$$|G(\xi) - G_0^{(0)}| = \min. \tag{15}$$

(the s. c. "homogeneous approximation"), and in the second case

$$\int_{1}^{+1} |G(\xi) = G_0^{(1)}|\, d\xi = \min. \tag{16}$$

(cf. [8], p. 63) in respective classes of functions (i.e. to the fourth or to the sixth order in the Tchebyshev sense). Integrally homogeneous coils are the best when objects of investigations are large, but small oscillations of the field are not very harmful (adiabatic demagnetization, Hall effect

*) It is easily seen that for coils determined by three shape parameters a higher tangency than the sixth is incompatible with [8]. The corresponding remark is also true for (ii).

**) This interval has been chosen for its simplicity. Any other may be considered on the same lines.

experiments?). The parameter p determines the weight function: for $p=\frac{1}{2}$ it is a constant, for $0 \leqslant p < \frac{1}{2}$ it enlarges the function under the integral sign in 14 near the ends of the interval (—1, 1), for $p > \frac{1}{2}$ the latter is diminished. Accordingly, the minimalization of the square deviation $[G(\xi)-G_0^{(p)}]^2$ is for $p < \frac{1}{2}$ stronger near the points $\xi = \pm 1$, for $p > \frac{1}{2}$ near the points $\xi = 0$. If $p \to \infty$ the integraly homogeneous (superhomogeneous) coil goes over into the locally homogeneous (superhomogeneous) one, i.e. the case (i) is a limiting case of (ii), and we see that our discussion is complete.

Because of the transcendental character of the function (5), the practical solution of the presented problems is difficult and may be carried out only by numerical methods. We present here only the solution of the first problem, as being the simplest but perhaps the most important case. One of us (J. M.) has found by usual numerical analysis that the locally homogeneous magnet has the following shape parameters (cf. Fig. 1)

(17) $$\alpha = 3.50, \quad \beta = 2{,}80, \quad \gamma = 0.116,$$

with an accuracy of about 1 per cent (note: the solution is most sensitive for the accuracy in γ). Furthermore,

(18) $$G_0 = 0.1314, \quad 2\,\xi_0 = 0.990,$$

where $2\,\xi_0$ is a "1 per cent decrease distance", a conventional criterion of hemogeneity often used by experimenters (cf. [5]). This coil may be regarded as a finite case corresponding to the infinitely thin Helmholtz magnet, because it has the same sort and quality of homogeneity in the middle. The Helmholtz magnet has, however, a much worse $2\,\xi_0$-distance

(19) $$2\,\xi_0 = 0.628.$$

For comparison we give also data for the magnet optimum only with respect to the field, i.e. satisfying condition (8) without (9) and (10), cf. [2] (units!),

(20) $$\alpha = 3, \quad \beta = 2, \quad \gamma = 0, \quad G_0 = 0{,}142, \quad 2\,\xi_0 = 0.3$$

We see that our magnet has a threefold better homogeneity in the $2\,\xi_0$-measure, by diminishing the central field by about 7.8 per cent. It is interesting also to compare our results with data obtained by Kurti [5] by the empirical "trial and error" method. He used the magnet (20) as a starting point and making a gap in the coil obtained the best results for

(21) $$\alpha = 3, \quad \beta = 2, \quad \gamma = 0{,}2, \quad G_0 = 0.128, \quad 2\,\xi_0 = 0.75,$$

i.e. only $2^1/_2$-fold amelioration of the homogeneity by 10 per cent, diminishing of the field. Our results are better by about 20 per cent because parameters α and β were also changed.

Further details will be published separately in Zeszyty Naukowe Uniwersytetu Wrocławskiego (Ser. B., Matematyka, Fizyka, Astronomia, III).

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EXPERIMENTAL PHYSICS

Optical and Photoelectric Properties of Thin CdSe Layers

by

W. WARDZYŃSKI and W. GIRIAT

Presented by W. RUBINOWICZ on February 18, 1960

CdSe is a semiconductor compound characterized by a very pronounced photoelectric effect [1], [2] presenting certain interesting properties [3]—[5]. Thin layers obtained by evaporation *in vacuo* are of advantage in the investigation of some semiconductor properties, especially in evaluating the influence of surface recombination on the photoelectric effect. Notwithstanding the fact that results of measurements on thin layers are more difficult to asses and that error is more considerable than in measurements on monocrystals, investigation of the properties of thin layers is of high interest in a number of applications.

The present investigation aimed to obtain information on the properties of thin layers of CdSe.

Preparation of samples

The semiconductor compound CdSe was prepared by way of direct synthesis of its elements. To this purpose, Cd and Se of impurities content not exceeding $10^{-5}\%$ were used. Pure Cd and Se were obtained from technical material by distillation *in vacuo* and zone melting [6]. The compound was synthesized in a sealed quartz tube under a pressure of about 10^{-6} mmHg. CdSe layers were prepared by evaporation *in vacuo* ($p = 10^{-4}$ mmHg) onto glass substrates at room temperature. The thickness of the layers was determined by weighing and by the optical method of interference, both methods yielding results that agreed satisfactorily. The layers were 0.5 to 10 μ thick. The electrodes were made of aquadag.

Measurement of optical and photoelectric properties

The layers thus obtained were used for measuring transmission *versus* wavelength at room temperature and at that of liquid nitrogen. Measurements were carried out with a glass monochromator. The light source was provided by a d. c. heated tungsten spiral of known spectrum distribution. The light incident on the sample was modulated mechanically. The output signal was transmitted to a photocell and amplified with a phase detection amplifier.

The same measuring device was used for photoelectric measurements. These were carried out with illumination modulated throughout the range of 150 to 1000 cps and at constant illumination. In the former case, the photoelectric current was measured with the phase detector, whereas a galvanometer was used in the latter. The samples were placed in air during the measurements.

Experimental results: optical properties

Fig. 1 shows the results of measurements of the transmission (P). In the long-wave range (0.8—2.5 μ), interference maxima appear. The dependence of the distance between consecutive maxima $\Delta 1/\lambda$ on the layer thickness d and refractive index n is accounted for by the formula

$$\Delta \frac{1}{\lambda} = \frac{1}{2\,dn}. \tag{1}$$

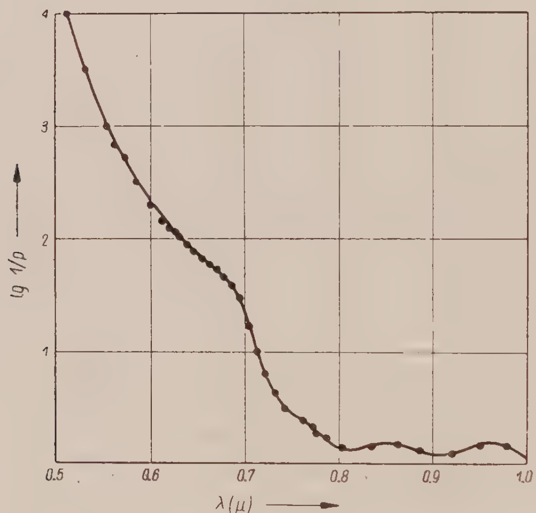

Fig. 1. Lg $1/p$ *versus* the wavelength

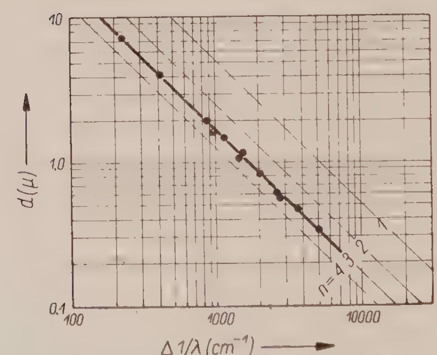

Fig. 2. Determination of the refractive index n.

Eq. (1) is represented on a logarithmic graph by a family of parallel straight lines corresponding to different values of n (Fig. 2). The value of $\Delta\, 1/\lambda$ was measured for a given thickness of the layer, and results plotted in Fig. 2. The experimental points coincide with the straight line corresponding to $n = 3$.

The usual formula [7] cannot be applied for computing the absorption coefficient α and the coefficient of reflection R, as the CdSe layer was evaporated onto glass substrates and hence its optical symmetry is impaired. The formula for the transmission of such a system, accounting for all multiple reflections, takes the form

$$(2)\quad P=\frac{J}{J_0}=(1-R)(1-R_1)(1-R_2)\times$$
$$\times D_1 D_2 \frac{1}{1-RR_1 D_1^2\, 1-D_2^2 R_2\,[R_1+R(1-R_1)^2 D_1^2]\,\dfrac{1}{1-RR_1 D_1^2}},$$

where R denotes the coefficient of reflection of the CdSe layer in respect to air,

R_1 — the coefficient of reflection at the CdSe/glass boundary,

R_2 — the coefficient of reflection of glass in respect to air,

$D_1 = e^{-\alpha_2 d_1}$,

$D_2 = e^{-\alpha_2, d_2}$,

α_1 — the absorption coefficient of the CdSe layer,

α_2 — the absorption coefficient of the glass substrate,

d_1 — the thickness of the CdSe layer,

d_2 — the thickness of the glass layer.

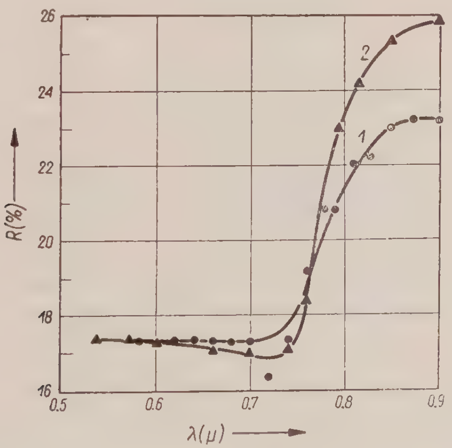

Fig. 3. Coefficient of reflection R *versus* the wavelength λ

With the known values of α_2, d_2 and R_2 and the experimental value of the transmission P for layers of different thickness d_1, the dependence of the coefficient of reflection R on the wavelength λ was computed from Eq. (2). The result is shown in Fig. 3 (graph 1). In order to check the dependence of R on λ the value of R was measured directly by reflection.

This was done for two angles of incidence (the angle subtended by the incident light beam and the line normal to the surface was 5° and 50°, respectively): in either case results were in good agreement. The values of R thus measured are assembled in Fig. 3 (graph 2). The graph is in arbitrary units and is normed to the value of graph 1 to 0.6 μ. Both graphs

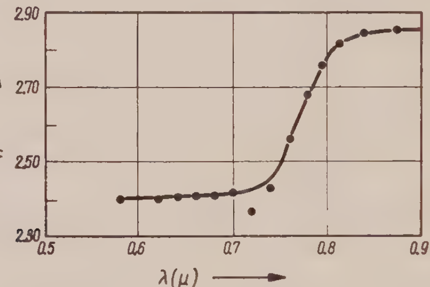

Fig. 4. Refractive index n *versus* the wavelength

are seen to be of the same shape. Fig. 4 brings the refractive index n *versus* λ as computed from Eq. (3). The position of the absorption edge at room temperature (graph 2) and at that of liquid nitrogen (graph 1) is shown in Fig. 5.

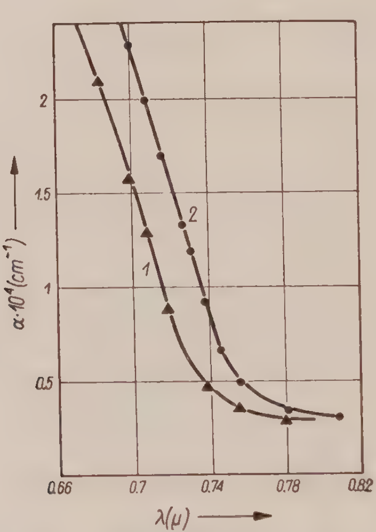

Fig. 5. Temperature shift of absorption edge:
graph 1 — temperature of liquid nitrogen, graph 2 — room temperature.

Experimental results: photoelectric properties

Fig. 6b shows graphs of distribution of the photoelectric current *versus* the wavelength for layer of different thickness and for a single crystal referred to equal incident energy. All the graphs are normed to the same

value of the maximum. The absorption coefficient is shown in Fig. 6a. The photoelectric current *versus* the light modulation frequency for layers of 1.2 μ and 10 μ is shown in Fig. 7. The effect of the photoelectrically exciting light frequency is different in thick and thin layers. In thick layers the distribution curves of the photoelectric current at $\lambda > \lambda_0$ (λ_0 denotes the position of the photoelectric current maximum) are inde-

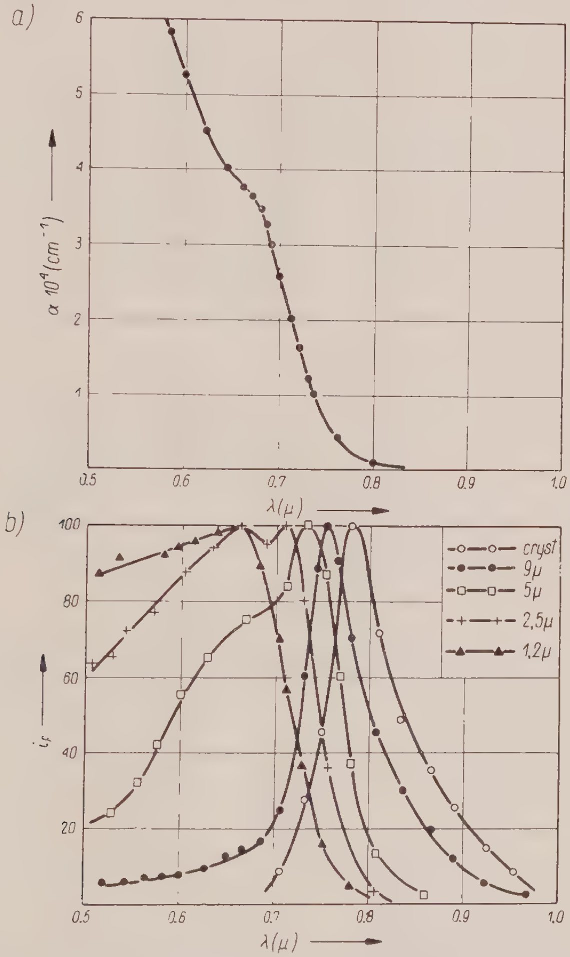

Fig. 6a. Absorption coefficient α, Fig. 6b. Photoelectric current λ *versus* the wavelength intensity *versus* the wavelength λ for CdSe layer of different thickness

pendent of the frequency throughout the range of 150 to 1000 cps. whereas at $\lambda < \lambda_0$ the latter is seen to influence the results (curves 3 and 4). Then continuous illumination is applied, the photoelectric current distribution curve for $\lambda > \lambda_0$ is displaced towards longer wave-lengths, and for $\lambda < \lambda_0$ a fall in the current is seen to occur (curve 5). In thin layers the photoelectric current is independent of the frequency throughout the range of

150 to 1000 cps (curve 1). With continuous illumination, the photoelectric current distribution curve shifts towards shorter wavelengths (curve 2). It is interesting to note the resemblance of curve 2 and that of the absorption coefficient *versus* the wavelength, as shown in Fig. 6a.

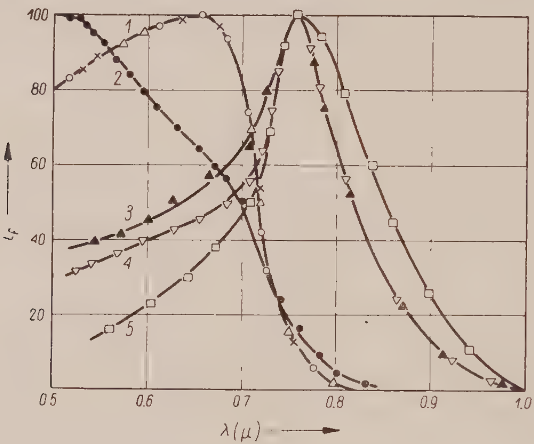

Fig. 7. Photoelectric current intensity *versus* the incident light modulation frequency, for layers 1,2 μ (graphs 1 and 2) and 10 μ (graphs 3 and 4) thick: graph 1 — incident light modulation frequency from 150 to 1000 cps,
graph 2 — constant illumination,
graph 3 — incident light modulation frequency 1000 cps,
graph 4 — frequency 150 cps,
graph 5 — constant illumination.

The photoelectric current distribution for a sample 10 μ thick, at room temperature and at that of liquid nitrogen is shown in Fig. 8.

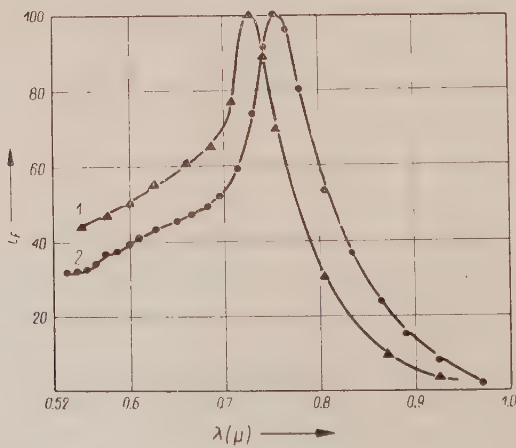

Fig. 8. Photoelectric current intensity *versus* the wavelenth λ at temperature of liquid nitrogen (graph 1) and at room temperature (graph 2).

Discussion of results

Results obtained for the refractive index, $n=3$, and for the coefficient of refraction R in the range of wavelengths exceeding 0.8 μ i. e. the range, wherein n was measured, should satisfy the equation

$$R=\frac{(n-1)^2+k^2}{(n+1)^2+k^2}, \tag{3}$$

with $k=\frac{\lambda a}{4\pi}$. Within the range of wavelengths considered, k^2 is small and may be neglected. By Eq. (3), the value of R should amount to 25%, and thus the experimental value of 23.2% yields satisfactory agreement with respect to the low degree of accuracy in determining n. Moreover, adsorption of gases may have been a factor lowering the experimental value of R.

At the absorption edge a maximum appears whose origin is obscure. The width of the forbidden zone is not determined unequivocally. It was assumed that the absorption edge is determined by the wavelength corresponding to the value, at which the linear segment of the coefficient absorption intersects the axis of abscissae. This leads to a value of $\Delta E = 1.64$ eV. The temperature shift of the absorption edge yields $\frac{d\Delta E}{dT} = 2.8\times10^{-4}$ eV/°C. Either value is lower than the respective one of [8] as obtained from photoelectric measurements.

It is seen from Fig. 6b that the spectral distribution of the photoelectric current intensity depends on the thickness of the layer. As the thickness diminishes, the maximum shifts towards shorter wavelengths. Also, the photoelectric current intensity throughout the range of short wavelengths is the greater, the smaller the layer thickness.

The foregoing results are explained, if we take volume and surface recombination into account. Generally, the photoelectric current intensity is a growing function of the absorbed energy and a falling function of the recombination. Volume recombination diminishes as the thickness of the layer increases, giving rise to current carriers; thus the intensity of

No.	Thickness of layer (d)	Position of maximum of photoelectric current $(\lambda_0)_{2^\circ}$	$d\alpha(\lambda_0)$
1	9×10^{-4} cm	0.756	4.5
2	6×10^{-4} cm	0.735	5.5
3	2.5×10^{-4} cm	0.710	5.0
4	1.2×10^{-4} cm	0.665	4.44

the photoelectric current will rise with the energy absorbed and with the thickness of the layer. Let λ_0 denote the wavelength for which $d\alpha(\lambda_0)\sim1$. The energy absorbed is the total incident energy of wavelength shorter than λ_0. At longer wavelengths, part of the radiation incident is transmitted and the amount of the energy absorbed diminishes.

The thickness of the layer, wherein absorption occurs, increases towards longer wavelengths of the incident radiation. For $\lambda > \lambda_0$, absorption occurs throughout the entire thickness of the layer. It follows that the maximum of the photoelectric current will lie at λ_0. For sufficiently thin layers (in the present investigation — layers of less than $1\,\mu$), the effect of surface recombination predominates. In such cases photoelectric current for wavelengths of less than λ_0 should exhibit the same value, as for λ_0, since recombination no longer depends now on the thickness of the layer, wherein current carriers are generated. This point of view is largely corroborated by the dependence of the photoelectric current on the incident wavelength for layers of different thickness as shown in Fig. 6a. The values of $d_\alpha(\lambda_0)$ for different values of the layer thickness are also given.

The effect of the exciting light frequency on the photoelectric current distribution (Fig. 7) is also related to the processes of recombination; however, the interpretation of experimental results presents various difficulties.

The authors avail themselves of this opportunity to express their thanks to Professor L. Sosnowski for his kind interest, valuable discussions and remarks, as well as to J. Mycielski Ma. Sc., J. Raułuszkiewicz Ma. Sc. for their discussion of the results.

Summary

The refractive index, coefficient of reflection and absorption edge were measured for the CdSe layers prepared by evaporation *in vacuo*. The distribution of the photoelectric current as a function of the incident wavelength for layers of different thickness, and the variations of these distributions as dependent on the temperature and modulation frequency of illumination, were determined.

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EXPERIMENTALE PHYSIK

Über den Mechanismus der Schwefelung von Cu-Zn-Legierungen mit Niedrigem Zn-Gehalt

von

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Vorgelegt von H. NIEWODNICZAŃSKI am 1. März, 1960

In den früheren Arbeiten [1], [2] haben die Verfasser bewiesen, dass die Schwefelung von Kupfer in flüssigem Schwefel bei 444°C ausschliesslich infolge der nach aussen gerichteten Diffusion des Metalls zustande kommt. Die Tatsache, dass unter diesen Bedingungen ein zweischichtiger Zunder entsteht, ist nicht durch die gleichzeitige Diffusion beider Reagenten zu erklären, wie es bisher angenommen wurde [3], [4]; sie ist vielmehr als Ergebnis der sekundären Prozesse anzusehen, die innerhalb der Zunderschicht infolge der Bildung eines Hohlraums zwischen dem Metall und dem Zunder stattfinden [5]. Bei der Schwefelung von Cu-Zn-Legierungen, die 5 bis 15 At % Zn enthalten, wird zwar auch ein zweischichtiger Zunder gebildet [6], der Mechanismus dieses Prozesses unterscheidet sich jedoch stark von demjenigen, der bei der Schwefelung von reinem Kupfer unter sonst gleichen Bedingungen festgestellt werden konnte. Unter Verwendung von radioaktiven Schwefelisotop ^{35}S haben die Verfasser bewiesen, dass der Bildungsprozess des an der Oberfläche erwähnter Liegierungen entstehenden zweischichtigen Zunders durch die gleichzeitige Diffusion von Kupfer und Schwefel bedingt wird [7].

Die äussere Zunderschicht besteht ausschliesslich aus Kupfer (I)-sulfid mit einem Kupferdefizit (annähernd $Cu_{1,8}S$); sie wird infolge der nach aussen gerichteten Diffusion von Kupfer gebildet. Die innere Schicht, die aus einem heterophasigen Gemisch von Kupfer- und Zinksulfide besteht [6], wird dagegen entweder vorwiegend oder sogar ausschliesslich infolge der kernwärts gerichteten Diffusion von Schwefel gebildet. Durch mikroskopische, röntgenographische und chemische Untersuchungen der äusseren, auf reinem Kupfer und auf den Cu-Zn-Legierungen in flüssigem Schwefel bei 444°C gebildeten, Zunderschicht wurden keine bemerkbaren Unterschiede in der Struktur und in der Zusammensetzung des gebildeten Kupfer (I)-sulfids [6], [8] festgestellt. Da bei der Schwefelung von reinem Kupfer die kernwärts gerichtete Diffusion des Schwefels durch die Zun-

derschicht überhaupt nicht auftritt und da im Falle der Schwefelung von Cu-Zn-Legierungen der Anteil der kernwärts gerichteten Diffusion von Schwefel im Wachstumprozess der inneren Zunderschicht deutlich zu beobachten ist, wurde als Arbeitshypothese angenommen, dass in der äusseren, auf erwähnten Legierungen entstehenden, Zunderschicht submikroskopische Spalten auftreten, durch die — in dieser Schicht — die kernwärts gerichtete Diffusion von Schwefel zustande kommt. Die Bildung dieser Spalten in der äusseren, auf Legierungen entstandenen, Zunderschicht wird durch einige noch nicht erforschte Ursachen bedingt.

Die oben angeführten Untersuchungsergebnisse zeigen daraufhin, dass im Bereich der Zn-Konzentration zwischen 0 und 5 At⁰/₀Zn eine Änderung des Mechanismus der Zunderbildung zu erwarten ist. Änderung beruht auf dem Ausbleiben der Spaltenbildung, wodurch der Anteil der kernwärts gerichteten Diffusion von Schwefel im Bildungsprozess der inneren Schicht stark herabgezetzt wird und keine Rolle mehr spielt. In der vorliegeden Arbeit haben die Verfasser den Versuch unternommen den Schwefelungsmechanismus im Falle der 1—4 At⁰/₀Zn enthaltenden Cu-Zn-Legierungen zu untersuchen, zwecks Feststellung, ob die erwartete Änderung des Mechanismus der Zunderbildung tatsächlich vorliegt, und zwecks Feststellung, ob die erwartete Änderung des Mechanismus der Zunderbildung tatsächlich vorliegt, und zwecks Erklärung, wie der Charakter dieser Änderung von der Zusammensetzung der Legierung abhängt.

Die Untersuchungen wurden unter Verwendung von radioaktivem Schwefelisotop ^{35}S durchgeführt. Die Legierungen wurden aus reinem Metallen (Kupfer — 99,98⁰/₀ Cu, und Zink — 99,95⁰/₀ Zn) in Graphittiegeln unter Decksalzen hergestellt. Die erhaltenen Abgusstücke wurden dann bis zur Stärke 0,4 cm kaltgewälzt und zwecks besserer Homogenisierung eine Stunde lang bei 500°C getempert. Die Oberfläche wurde dann abgeschabt und mit Schmirgelpapier poliert. Aus den auf diese Weise hergestellten Präparaten wurden Platten $4{,}0 \times 3{,}0 \times 0{,}3$ cm ausgeschnitten. Vor der Schwefelung wurde die Oberfläche jedes Präparates leicht mit verdünnter HNO_3 — Lösung (1 : 1) ca 10 Sekunden lang geätzt. Die Methode, sowie auch die Durchführung der Messungen, sind in früheren Arbeiten beschrieben worden [1], [7]. Die Schwefelung wurde im flüssigen Schwefel bei 444°C durchgeführt. Die Legierungsplatten wurden zuerst in einem nichtaktiven Medium so lange vorgeschwefelt bis sich ihre Oberfläche mit einer Zunderschicht bedeckt hat, derer Stärke die vollständige Absorption der ^{35}S β-Strahlung sicherstellte. Nachher wurde in die Reaktionskammer eine aus radioaktivem Schwefel verfertige Pastille eingeführt und die Schwefelung verschieden lang fortgesetzt. Die Dauer der Vorschwefelung wurde so gewählt, dass unabhängig von der Zusammensetzung der Legierung, die Stärke der äusseren Zunderschicht immer den Wert 5.10^{-2} cm überstieg. Diese Stärke ist ungefähr zweimal grösser als die kritische Zunderschichtstärke, die die vollständige Absorption der ^{35}S-Strahlung

gewährleistet. Der Schwefelungsprozess wurde durch Herausnahme des Präparates aus dem Reaktionsmedium und Verbrennung des an der Oberfläche anhaftenden Schwefelrestes unterbrochen. Die Dauer der Verbrennung der Schwefel an der Oberfläche des Präparates schwankte zwischen 5 und 10 Sekunden. Da die Schwefelung 8—40 Minuten dauerte, kann in der ersten Annähherung angenommen werden, dass der Vorgang momentan unterbrochen wurde. Nach der Beendigung der Schwefelung und nach der Abtrennung des Zunders von der äusseren sowie auch der inneren Seite der Zunderschicht, die an den beiden grösseren Oberflächen der Legierugsplatte gebildet worden ist, wurden Messungen vorgenommen. Die Ergbnisse der Aktivitätsmessungen sind in der Tafel I zusammengestellt.

Es folgt aus den in der Tafel zusammengestellten Daten, dass die Diffusion von Schwefel — unabhängig von der Zusammensetzung der untersuchten Legierung — am Bildungsprozess des Sulfidzunders teilnimmt. Auf Grund der erhaltenen Ergebnisse lässt es sich jedoch ausserdem schliessen, dass der Anteil der kernwärts gerichteten Diffusion von Schwefel im Prozess der Zunderbildung mit abnehmendem Zn-Gehalt in der Legierung auch abnimmt und dass dieser Anteil im Falle einer 1,3 At%Zn enthaltenden Legierung schon gar unbedeutend ist. Diese Schlussfolgerung scheint im Lichte der in der letzten Spalte der Tafel zusammengestellten Daten völlig berechtigt. In dieser Spalte sind die Quotienten der an der äusseren und an der inneren Oberflächen gemessenen Aktivitäten in Prozenten angegeben. Im Falle der 4,4 At%Zn enthaltenden Legierungen, bei dem Zeitverhältnis t_2/t_1 gleich 9, beträgt die Aktivität der inneren Zunderschichtoberfläche 43% der Aktivität der äusseren Oberfläche, während die Aktivität der inneren, bei gleichem Schwefelungszeitverhältnis auf 1.3%-tiger Legierung gebildeten Zunderschicht, lediglich 1,9% der Aktivität der äusseren Zunderschichtoberfläche beträgt. Die erhaltenen Resultate zeigen also daraufhin, dass der Mechanismus der Schwefelung von Legierungen, die weniger als 2 At%Zn enhalten, anders ist als der Mechanismus der Sulfidzunderbildung im Falle von Legierungen die über 4% Zn enthalten. Aus den morphologischen Untersuchungen des Zunders geht hervor, dass Wachstummechanismus der inneren Zunderschicht — im Falle von Legierungen die weniger als 2 At% enthalten — dem Mechanismus der Bildung der inneren Zunderschicht im Falle von reinem Kupfer analog ist [5].

Es kann also angenommen werden, dass das Wachstum der inneren Schicht des Zunders im Falle von untersuchten Legierungen vorwiegend dank der Zersetzung der äusseren Zunderschicht erfolgt. Diese Zersetzung findet an der Phasengrenze zwischen der inneren und äusseren Schicht statt und führt zur Bildung von freiem Schwefel herbei, der mit der Kernoberfläche reagiert und die Bildung der inneren Schicht bedingt. Bei der

Schwefelung von Legierungen, die über [4] At%Zn enthalten, scheint im Wachstumprozess der inneren Zunderschicht die kernwärts gerichtete Diffusion von Schwefel die entscheidende Rolle zu spielen.

TAFEL

No	% Zn in Legierung	t_1	t_2	t_2/t_1	$A^{(a)}$	$A^{(i)}$	$\frac{A^{(i)}}{A^{(a)}} 100$
1	1,3	3	3	1	5031	49	0,9
2	,,	3	9	3	5167	74	1,4
3	,,	3	15	5	5209	176	3,4
4	,,	3	21	7	5060	193	3,8
5	,,	3	27	9	5675	110	1,9
6	1,9	4	4	1	4295	37	0,8
7	,,	8	8	1	778	4,5	0,6
8	,,	4	12	3	4471	56	1,2
9	,,	8	24	3	786	25	3,2
10	,,	4	20	5	4400	910	20,7
11	,,	7	35	5	674	112	16,7
12	,,	8	40	5	800	76	9,5
13	,,	4	28	7	4457	962	20,8
14	,,	4	28	7	670	101	15,1
15	,,	4	36	9	4591	680	14,9
16	2,4	4	4	1	3128	111	3,6
17	,,	4	12	3	3402	239	7,0
18	8	4	20	5	3265	582	17,9
19	,,	4	28	7	3341	623	18,7
20	,,	4	36	9	2837	862	30,2
21	4,4	4	4	1	3419	447	13,1
22	,,	4	12	3	2495	607	24,4
23	,,	6	18	3	6 8	193	28,0
24	,,	4	20	5	2577	1044	40,2
25	,,	6	30	5	547	159	29,1
26	,,	4	28	7	2462	1255	51,0
27	,,	4	36	9	2720	1170	43,0

t_1 — Dauer der Schwefelung im inaktiven Schwefel (in Minuten)
t_2 — Dauer der Schwefelung im aktiven Schwefel (in Minuten)
$A^{(a)}$ — Aktivität (Imp./Min.) der äusseren Schicht des Zunders an der Oberfläche, die während der Schwefelung im unmittelbaren Kontakt mit Schwefel stand
$A^{(i)}$ — Aktivität (Imp./Min.) der inneren Schicht des Zunders an der Oberfläche, die während der Schwefelung im unmittelbaren Kontakt mit Metall stand

Die in der Tafel zusammengestellten Daten ziegen daraufhin, dass die hier untersuchte Änderung des Mechanismus der Schwefelung im Falle von wenig Zn enthaltenden Legierungen mit der Änderung der Zn-Konzentration in der Legierung verbunden ist und stufenweisse im Konzentrationsbereich zwischen 1 bis 4 At%Zn erfolgt.

Es sind noch weitere Untersuchungen erforderlich, um den relativen Anteil einzelner Teilprozesse im gesamten Wachstumprozess der inneren Zunderschicht auf den Cu-Zn-Legierungen im erwährten Übergangsintervall der Zn-Konzentration in der Legierung bestimmen zu köonnen.

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БЮЛЛЕТЕНЬ ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

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А. ЛЕЛЕК, **О ТРИОДИЧЕСКОЙ ТЕОРЕМЕ МУРА** . . . . стр. 271—276

Пусть подмножество T пространства X называется τ-множеством в X если для некоторого $\varepsilon > 0$ и некоторых множеств $P, Q \subset T$ (из которых P будет называться *сердцевиной* τ-множества T) произведение $\overline{P \cap Q}$ связно и для всякой области $R \subset X$ условия $P \cap Q \subset R$ и $d(\overline{P \cap Q}, \overline{R}) < \varepsilon$ (где d — хаусдорффовое расстояние) влекут за собой следующее:

(i) компонента множества $P \cap R$, содержащая $P \cap Q$ разбивает область R,

(ii) компонента множества $Q \cap R$, содержащая $P \cap Q$ отлична от $P \cap Q$.

Если $A, B \subset X$, мы будем говорить, что A *пронизывает* B, если B разбивает некоторую область $R \subset X$ и, по крайней мере, одна из компонент множества $A \cap R$ пересекает две различные компоненты множества $R - B$.

Известная теорема Мура о триодах обобщена следующим образом:

ТЕОРЕМА. *Пусть* $\boldsymbol{T} = \{T_\iota\}_{\iota \in \Gamma}$ *— несчетное семейство (т.е.* $\aleph_0 < \overline{\overline{\Gamma}}$*) τ-множеств, содержащихся в локально связном и вполне ограниченном метрическом пространстве X и пусть сердцевины произвольных двух элементов $\boldsymbol{T}$ не пересекаются. Тогда существует несчетное семейство $\boldsymbol{T}' \subset \boldsymbol{T}$ и такое естественное его упорядочение $\prec$, что если $T_\iota, T_\varkappa \in \boldsymbol{T}'$ и $T_\iota \prec T_\varkappa$, то Q_ι пронизывает $P_\varkappa$.*

СЛЕДСТВИЕ. *Пусть $\{C_\iota\}_{\iota \in \Gamma}$ — несчетное семейство непересекающихся n-клеток евклидова пространства E^{n+1}. Тогда существует несчетное подсемейство $\{C_\iota\}_{\iota \in \Gamma}$ (т.е. $\Gamma_1 \subset \Gamma$ и $\aleph_0 < \overline{\overline{\Gamma_1}}$) такое, что любая внутренняя точка произвольной n-клетки C_ι, где $\iota \in \Gamma_1$, недостижима из множества $E^{n+1} - \bigcup_{\iota \in \Gamma_1} C_\iota$.*

И. СЛОМИНЬСКИЙ, **О СОВМЕСТНОМ ПОГРУЖЕНИИ АБСТРАКТНЫХ КВАЗИ-АЛГЕБР В ТОЖДЕСТВЕННО ОПРЕДЕЛИМЫЕ КЛАССЫ АБСТРАКТНЫХ АЛГЕБР** . . . . стр. 277—282

Работа является обобщением теории гомоморфизмов квази-алгебр в алгебры, принадлежащие к произвольному тождественно определимому классу, приведенной автором в работе [2].

Пусть $\boldsymbol{A}_t$, $t \in T$, — произвольное семейство квази-алгебр типа Δ, и пусть $\mathfrak{B}$ — произвольный класс алгебр типа Δ. Если h_t, для $t \in T$, является произвольным гомоморфизмом (изоморфизмом) квази-алгебры $\boldsymbol{A}_t$ в ту же алгебру $\boldsymbol{B} \in \mathfrak{B}$, то семейство $\boldsymbol{h} = \{h_t\}_{t \in T}$ называем совместным $\mathfrak{B}$-гомоморфизмом ($\mathfrak{B}$-изоморфизмом) квази-алгебр $\boldsymbol{A}_t$, $t \in T$ в алгебру $\boldsymbol{B}$.

Алгебру $\boldsymbol{B} \in \mathfrak{B}$ называем совместным $\mathfrak{B}$-расширением квази-алгебр $\boldsymbol{A}_t$, $t \in T$, если имеется общий $\mathfrak{B}$-изоморфизм квази-алгебр $\boldsymbol{A}_t$, $t \in T$, в алгебру $\boldsymbol{B}$. Пусть $\mathfrak{B}$-произвольный тождественно определимый класс алгебр типа Δ.

В пкт. 1 настоящей работы приводится теория совместных $\mathfrak{B}$-гомоморфизмов квази-алгебр $\boldsymbol{A}_t$, $t \in T$, а также даются условия необходимые и достаточные для того, чтобы существовало совместное $\mathfrak{B}$-расширение квази-алгебр $\boldsymbol{A}_t$, $t \in T$.

В пкт. 2 работы вводится понятие $\mathfrak{B}$-свободного произведения квази-алгебр $\boldsymbol{A}_t$, $t \in T$. $\mathfrak{B}$-свободное произведение квази-алгебр $\boldsymbol{A}_t$, $t \in T$, существует всегда и является с точностью до изоморфизма однозначно определенным через $\boldsymbol{A}_t$, $t \in T$. Затем определяется естественное $\mathfrak{B}$-свободное произведение квази-алгебр A_t, $t \in T$, которое в случае когда все квази-алгебры A_t являются алгебрами, совпадает с $\mathfrak{B}$-свободным произведением, введенным Р. Сикорским в работе [1]..

Кроме того, даются необходимые и достаточные условия для того, чтобы существовало естественное $\mathfrak{B}$-свободное произведение квази-алгебр $\boldsymbol{A}_t$, $t \in T$.

А. ГУЛЯНИЦКИЙ и С. СВЕРЧКОВСКИЙ, **МОЩНОСТЬ СЕМЕЙСТВА АЛГЕБР С ДАННЫМИ МНОЖЕСТВАМИ ЭЛЕМЕНТОВ** . . . стр. 283—284

Рассмотрим семейство всех алгебр (см. [1], vii), для которых множество S является множеством элементов. Е. Л. Пост доказал в своей работе [2], что если S обладает точно двумя элементами, то семейство всех алгебр, определенных на S имеет мощность $\aleph_0$.

Авторы приводят мощность семейства всех неизоморфных алгебр, определенных на S, когда S обладает более, чем двумя элементами.

Тогда справедливо следующее утверждение:

Если S является конечным множеством, обладающим более чем двумя элементами, либо $S = \mathfrak{m}$, $\mathfrak{m} \geqslant \aleph_0$ то мощность семейства всех неизоморфных алгебр, определенных на S равна континуум либо $2^{\mathfrak{m}}$. соответственно.

К. УРБАНИК и Ф. В. ВРАЙТ, **АЛГЕБРЫ С АБСОЛЮТНЫМ ЗНАЧЕНИЕМ** стр. 285—286

Рассматриваются алгебры с абсолютным значением над полем действительных чисел, т.е. вообще неассоциативные алгебры, являющиеся линейным нормированным пространством относительно мультипликативной нормы. Основным результатом работы является следующая теорема:

Любая алгебра с абсолютным значением, обладающая единицей, изоморфна либо полю действительных чисел, либо полю комплексных чисел, либо алгебре кватернионов, либо наконец алгебре Кэли-Диксона.

Пример бесконечномерной алгебры с абсолютным значением показывает, что предположение существования единицы существенно.

Г. КРЕЙЗЕЛЬ, **РЕЗУЛЬТАТЫ ВЫРАЖАЮЩИЕ НЕОДНОЗНАЧНОСТЬ ДЛЯ ТРАНСФИНИТНЫХ ПРОГРЕССИЙ** . . . . стр. 287—290

Автор рассматривает отобразования $y \to A_y$ или $y \to C_y$, где A_y является рекурсивно перечислимым семейством множеств (т. е. $n \epsilon A_y \equiv (Et) R(y; n, t)$, R — рекурсивное), а C_y — рекурсивно перечислимым семейством функций, т.е. $C_y = \{F_p : (m, n) \{\langle m, n \rangle \epsilon F_p \equiv S(y; p, n, m)\}\}$, S — рекурсивное, удовлетворяющее условию $(p)(n)(E!\, m)\, S(y; p, n, m)$. Автор предполагает, что y принимает значение из подмножества O^* множества O всех обозначений для конструктивных трансфинитных чисел, причем $z < O\, t \epsilon O^* \Longrightarrow z \epsilon O^*$.

Отобразования $y \to A_y$, $y \to C_y$ автор называет прогресивными, если из $y < O\, z$ следует, что A_y является собственным подмножеством A_z, а C_y — собственным подмножеством C_z.

Теорема доказанная в работе утверждает, что если O^* — замкнутое относительно некоторых операций, рассматриваемых в доказательстве и $A_y = A_z$ для всех $y, z \epsilon O^*$ таких, что $|y| = |z|$, то $|y| < \omega^3$ для $y \epsilon O^*$; и, подобным образом, если $C_y = C_z$ для всех $y, z \epsilon O^*$ таких, что $|y| = |z|$, то $|y| < \omega^4$ для $y \epsilon O^*$.

Э. МАРЧЕВСКИЙ и К. УРБАНИК, **АБСТРАКТНЫЕ АЛГЕБРЫ, В КОТОРЫХ ВСЕ ЭЛЕМЕНТЫ НЕЗАВИСИМЫ** . . . . стр. 291—293

В работе доказывается, что существуют точно три невырожденные алгебры, состоящие из независимых элементов в смысле работы [1].

Т. ВАЖЕВСКИЙ, **О ДИФФЕРЕНЦИРУЕМОСТИ ПРЕДЕЛА ПОСЛЕДОВАТЕЛЬНОСТИ ФУНКЦИЙ, ОБЛАДАЮЩИХ ОДНОСТОРОННЕЙ ПРИБЛИЖЕННОЙ ПРОИЗВОДНОЙ (ДЛЯ СЛУЧАЯ ПРОСТРАНСТВА БАНАХА)** стр. 295—299

Автор приводит обобщение соответствующей классической теоремы для случая последовательности функций, обладающих односторонней приближенной производной.

Значения рассматриваемых функций действительной переменной лежат в пространстве Банаха.

П. БАЖЕВСКИЙ, **О СУЩЕСТВОВАНИИ И ОДНОЗНАЧНОСТИ ИНТЕГРАЛОВ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ДЛЯ СЛУЧАЯ ПРОСТРАНСТВА БАНАХА** . . . . стр. 301—305

Автор доказывает (не пользуясь аксиомой выбора), что некоторое введенное Е. Камке условие однозначности интегралов обыкновенных дифференциальных уравнений, рассматриваемых в пространстве Банаха, является также достаточным условием для их существования.

А. ШИНЦЕЛЬ, **О КОНГРУЭНЦИИ** $a^x \equiv b \pmod p$ . . . стр. 307—309

В работе доказана следующая теорема:

Если a, b — целые рациональные числа, $a > 0$ и $b \neq a^k$ (где k — целое рациональное), то существует бесконечное множество таких простых рациональных чисел p, что конгруэнция $a^x \equiv b \pmod p$ не разрешима в рациональных целых числах x.

В доказательстве использована теория целых чисел полей деления круга.

Г. БРОВКИН и А. ШИНЦЕЛЬ, **ОБ УРАВНЕНИИ** $2^n - D = y^2$ стр. 311—318

В работе рассматривается уравнение $2^n - D = y^2$. Приводится метод решения этого уравнения в натуральных числах, если $D \equiv 7 \bmod 8$ или D — число Мерсенна или D имеет простой делитель формы $8t \pm 3$.

Нахождение решений уравнения при $D > 0$ и $D \equiv 7 \bmod 8$ сводится к исследованию некоторой рекуррентной последовательности (Теорема 3).

Приведены все решения уравнения $2^n - D = y^2$, найденные по этому методу при $0 < D \leqslant 150$.

Р. С. ИНГАРДЕН и Е. МИХАЛЬЧИК, **ОБ ОПТИМАЛЬНОЙ ОДНОРОДНОСТИ ПОЛЯ В СОЛЕНОИДАХ БОЛЬШОЙ МОЩНОСТИ** . . . . стр. 319—324

Дается определение локально и интегрально однородных соленоидов и вычислены параметры локально однородного прямоугольного соленоида с постоянной плотностью тока и со щелью перпендикулярной к оси симметрии.

В. ВАРДЗЫНЬСКИЙ и В. ГИРЬЯТ, **ОПТИЧЕСКИЕ И ФОТОЭЛЕКТРИЧЕСКИЕ СВОЙСТВА ТОНКИХ СЛОЕВ** CdSe . . . . стр. 325—332

Для тонких слоев CdSe, полученных испарением в вакууме, проведены измерения показателя преломления, коэффициента отражения и коэффициента поглощения. Установлено распределение фототока в зависимости от длины волны для разных толщин слоев, а также изменения этих распределений, вызванные влиянием температуры и частоты модуляции света.

В предыдущих своих работах авторы показали, что во время сульфидирования меди в жидкой сере (445°C) нарастание окалины происходит исключительно за счет наружу-направленной диффузии меди [1], [2]. Однако в случае сульфидирования сплавов меди с цинком (4,8 — 15% Zn) в этих же самих условиях реакции, обнаружено двухстороннюю диффузию меди и серы через окалину [7].

В работе авторы представляют результаты исследований механизма сульфидирования в жидкой сере низкопроцентных сплавов меди с цинком (1,3 — 4,4% Zn), проведеных по методу меченых атомов (S^{35}), которая применялась в предшествующих работах.

Обнаружено, что в таком случае тоже имеет место двухсторонняя диффузия компонентов, но степень участия диффузии серы в процессе образования окалины падает вместе с понижением концентрации цинка в сплаве.

TABLE DES MATIÈRES

Mathématique

Physique Théorique

Physique Expérimentale

Cena zł 20.—